Evaluating the Market Risk Premium

Statistical properties of the historic market risk premium

Report to the ERA on the 2022 draft gas rate of return instrument

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Client:

Economic Regulatory Authority Western Australia

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Consultants:

Dr Rohan Sadler and Alex Stuckey

email: rohan.sadler@pinklake.com.au

ACN: 611 093 120

ABN: 60 611 093 120

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Executive Summary

The Economic Regulation Authority engaged Pink Lake Analytics to provide advice on the statistical analysis of the historic market risk premium in response to submissions it has received for the 2022 Gas Rate of Return Instrument.

Competition Economists Group (CEG) provide the submission "*Estimating MRP for the ERA 2022 RoRI",* recommending changes to the calculation of the market risk premium (MRP) applied in the Authority's rate of return instrument.

The calculation of the geometric mean of observed excess returns can be considered in several ways. The CEG submission contends a method of calculation, recommending the difference between the geometric means of the market returns and geometric mean of the risk-free-rate returns (CEG method) over the geometric mean of the series of differences between the observed market return and risk-free rate return (the current ERA method). We show that in practice the CEG method results in larger geometric mean estimates, although the difference is very small. We do not see a statistical reason to favour one method of calculating the geometric over the other, although the ERA estimator (geometric mean of differences) has the desirable property that under the scenario where the annual excess returns are constant, the ERA estimator produces a geometric mean that matches these annual excess returns.

Currently, the ERA applies a weighted average of the arithmetic and geometric means of the observed excess returns. There is some conjecture as to the ideal weighting of the arithmetic and geometric means and CEG recommend an estimate that is purely the arithmetic mean. The decision on whether the arithmetic mean alone is an adequate estimator comes down to at least three factors: the uncertainty in the estimation of the arithmetic and geometric means as estimators of the MRP; the length of the investment horizon over which returns are considered relevant; and the presence of autocorrelation in the return series. The available literature shows theoretically that the geometric mean should receive increased weight as one or more of the following occur: the estimation span decreases; the investment horizon increases; or autocorrelation is present in the return series. For prediction of excess return at a single time period, the arithmetic mean is theoretically the optimal estimator of the MRP. CEG's submission and some cited textbooks use simple numerical examples to show that the arithmetic mean is preferable for horizons beyond one period also. However, these examples are a straw man that does not relate to the problem of estimating the MRP, are overly simplistic as they assume that there is no uncertainty in the estimation of the MRP and that there is no autocorrelation in the return series.

The literature is conclusive in that the arithmetic mean is upwardly biased for predicting future returns over multiple periods. A common method in the literature to address this upward bias is to use a *composite* estimator that is a weighted mean of the arithmetic mean and geometric mean. The question from a statistical perspective then becomes what weight should be assigned to the geometric mean. Invariably, the choice of weighting in the MRP estimator relates to the estimator which best minimises overall prediction error of future portfolio returns.

We conduct a simulation study to show the relative accuracy and precision of several estimators when log returns are simulated from a GARCH(1,1) model. This study shows the conditions under which the arithmetic mean of excess returns is and is not adequate for the estimation of the market risk premium. The study results echo the theoretical results in the literature that more weight should be given to the geometric mean as the horizon of investment increases.

Our scope of works requested we evaluate the appropriateness of holding period and horizon parameters. We cannot offer guidance from a statistical point of view on what the appropriate horizon should be. We see the choice of an appropriate horizon is a regulatory judgement, to be made in the context of the period of the regulatory agreement, the long-term nature of the utility assets, and other factors deemed important by the regulator.

1 Terms of Reference

1. Pink Lake Analytics was invited by the Authority to provide advice on the estimator for the historic market risk premium, focusing on:
   - The geometric mean formula proposed by CEG compared with the one currently used by the ERA, noting if there are any biases when they are compared.
   - The statistic properties of the historic market risk premium conducted by Dr Lally.
2. As such, the following documents in support of the Submission have been reviewed:
   - Competition Economists Group – *Estimating MRP for the ERA 2022 RoRI* – August 2022
   - Dr Martin Lally – *Tests of Mean Stationarity for Australian Share Market Returns Data* -June 2022
   - Pink Lake Analytics – *Estimation of the Market Risk Premium: A review of weighting arithmetic and geometric means* – December 2017 (Attachment 2). https://www.erawa.com.au/cproot/18943/2/WPAA4%20-%20Estimation_of_the_Market_Risk_Premium.PDF
3. The scope of this work includes:
   - Providing advice on how the arithmetic and geometric means could be used by the ERA as an estimator for the historic market risk premium to achieve its task of setting revenues. This advice would consider the relevant factors in a weighting/combination scheme for the two means through assessment of the weighting on the statistical properties of the market risk premium (MRP) estimator.
   - Considering the impact in terms of bias and predictive error of serial autocorrelation, volatility and non-stationarity of time series of market returns, alongside the form of the estimator of the geometric mean and sampling error, on the estimator of the MRP
   - Considering through the review of literature the impact of investment strategy, comparing an ‘invest and accumulate’ against a ‘long-term investor holding’ strategy reflecting the task of setting annual revenues through a rate of return, on how the arithmetic and geometric means may be weighted.
   - Evaluating the appropriateness of holding period and horizon parameters if the reviewed literature suggesting that a weighting of the arithmetic mean and geometric mean are relevant.

Introduction

4. The ERA is undertaking two projects related to the determination of the WACC for regulatory purposes. These WACC determinations will impact the regulated revenues of the energy networks the ERA regulates. These WACC projects are for:

   - The 2022 Gas Rate of Return Instrument (Gas Rate of Return Instrument) review, under the National Gas Law and National Gas Rules. The Gas Rate of Return Instrument will apply to the access arrangements of ATCO's Mid-West and South-West Gas Distribution Systems, Goldfields Gas Transmission Pty Ltd's Goldfields Gas Pipeline, and DBNGP (WA) Transmission Pty Ltd's Dampier to Bunbury Natural Gas Pipeline.

   - The WACC determination for Western Power's fifth access arrangement, under the *Electricity Networks Access Code 2004*. Western Power is Western Australia's largest electricity network and provides electricity to customers throughout the South West.

5. On 17 July 2022 the ERA published a draft instrument and explanatory statement for the Gas Rate of Return Instrument. The explanatory statement set out the ERA's proposed methods for calculating the allowed rate of return on capital for gas network service providers and invited stakeholder submissions.

6. The Independent Panel reviewed the Draft gas instrument and explanatory information, with their report published on 24 August 2022. Submissions in response to the draft instrument and the Independent Panel's report were due on 6 September 2022. Six public submissions were received. The draft instrument, explanatory statement and all public submissions can be accessed on the ERA's website[1].

7. Of these submissions, the Australian Gas Infrastructure Group commissioned the Competition Economists Group (CEG)[2] to provide advice on the historic market risk premium. This advice proposes an alternative formula to calculate the geometric mean, along with the weights that should be provided to the arithmetic and geometric mean.

8. Furthermore, a Dr Lally report[3] concluded that the population means for excess returns, nominal returns and real returns in the Australian share market are the same in all of the years 1883 – 2021, i.e., mean stationarity prevails in the times series of financial data used to prepare estimates of the market risk premium.

[1] 2022 Gas Rate of Return Instrument Review. https://www.erawa.com.au/gas/gas-access/guidelines/gas-rate-of-return-instrument/2022-gas-rate-of-return-instrument-review.

[2] Competition Economists Group – *Estimating MRP for the ERA 2022 RoRI* – August 2022. https://www.erawa.com.au/cproot/22869/2/AGIG-submission-to-draft-gas-instrument---CEG-report-on-MRP---to-publish.PDF.

[3] Dr Martin Lally – *Tests of Mean Stationarity for Australian Share Market Returns Data* -June 2022. https://www.aer.gov.au/system/files/Lally - Tests of mean stationarity for Australian share market returns data_0.pdf

2 Review of the documents under consideration

2.1 Competition Economists Group – *Estimating MRP for the ERA 2022 RoRI*, August 2022

2.1.1 Calculation of the geometric mean

9. Competition Economists Group (CEG) argue that:[4]

> *"the geometric mean excess return over the risk free rate should be estimated as the difference between the geometric mean return on the market portfolio and the geometric mean return from investing in the risk free rate over the same estimation window. By contrast, the ERA's method was to derive an annual estimate of the MRP (by subtracting an annual risk free rate from the annual market return) and then to compound the annual estimate of the MRP."*

10. CEG then reason that:[5]

> *"In any event, the ERA is not estimating a risk premium; being the excess return above the return from investing in the risk free rate which is the definition of the market risk premium in the CAPM. Rather, the ERA is estimating the absolute return on a risky strategy funded 100% by borrowing at the risk free rate. These are not the same concepts and, hence, they do not give the same answers (at least not in all circumstances)."*

11. In support of their position CEG cite Dimson, Marsh and Staunton for Credit Suisse:[6]

> *"Investors expect a reward for exposure to risk. This is the risk premium, which we measure relative to the returns on Treasury bills and, if appropriate, government bonds. To do this, we estimate the geometric difference between the realized return on an asset and the risk-free rate of return that was available over the same period."*

12. CEG state that:[7]

> *"Indro and Lee (1997) also compound returns (not risk premiums) (…) To the best of our knowledge the only other organisation that adopts the ERA's approach is the AER – although the AER does not explicitly rely on the geometric average MRP. "*

13. The AER has stated in the past that a 10-year market risk premium may be approximated by the geometric average of annual MRPs:[8]

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[4] Competition Economists Group – *Estimating MRP for the ERA 2022 RoRI* – August 2022, paragraph 5, page 4.

[5] Competition Economists Group – *Estimating MRP for the ERA 2022 RoRI* – August 2022, paragraph 13, page 6.

[6] Competition Economists Group – *Estimating MRP for the ERA 2022 RoRI* – August 2022, paragraph 14, page 6, citing Dimson, Marsh and Staunton - *Credit Suisse Global Investment Returns Yearbook, Summary edition* - 2021 page 23.

https://www.credit-suisse.com/media/assets/corporate/docs/about-us/research/publications/credit-suisse-global-investment-returns-yearbook-2021-summary-edition.pdf

[7] Competition Economists Group – *Estimating MRP for the ERA 2022 RoRI* – August 2022, paragraph 16, page 6.

[8] For example, AER - *Draft Distribution Determination Aurora Energy Pty Ltd 2012–13 to 2016–17* - November 2011, pages 228-229

"Historical data, on the other hand, is usually presented in terms of annual returns and annual MRPs. However, a 10 year MRP can be approximated from annual MRPs by determining a geometric average of ten annual MRPs within that 10 year period. This geometric average approximates the 10 yearly MRP in annual terms."

14. Recent work by Kaserer[9] compares different estimators commonly applied to estimating unbiased discount rates. These estimators applied directly to excess returns within a simulation study, before being applied to estimation of the market risk premium for real data. These estimators were derived either as arithmetic means, geometric means, mean of arithmetic means and geometric means, one of four Cooper estimators[10] (including that of Indro and Lee[11]), the estimator of Elsner and Krumholz[12] and the estimator of Blume[13]. Their method of applying the geometric mean to the excess returns to estimate the market risk premium matches that of the Authority, in that Kaserer "report the statistical distribution of the continuously compounded market risk premium" using the arithmetic and geometric estimators of the excess returns.

15. The key point of difference between the CEG approach and that of the Authority is largely statistical. The Authority seeks an unbiased estimator of the market risk premium to be applied to the time horizon of a long-term holding.

16. In contrast, the proposal by CEG to estimate the market risk premium by calculating the difference between the geometric estimator to the market returns and risk-free rate does not fully examine the property of the estimators of the market risk premium for either bias or variance. Rather than *'having established the correct measure of the historical realized compound equity market risk premium'*[14] via rigorous statistical methods in their paper, CEG appear to have appealed simply to precedent practice. Indeed, even though Indro and Lee focus on the estimation of long-term market returns they do not view the arithmetic and geometric estimators as anything but a proxy for future returns:[15]

"Since risk premia are not constant … and can depend on the choice of measurement period, averaging method, or portfolio weighting …, how should the historical monthly market return data be used to compute the risk premium? In practice, the arithmetic and geometric average

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[9] Kaserer, Christoph. "Estimating the market risk premium for valuations: arithmetic or geometric mean or something in between?." *Journal of Business Economics* 92, no. 8 (2022): 1373-1415.

[10] Cooper, I., 1996. Arithmetic versus geometric mean estimators: Setting discount rates for capital budgeting. *European Financial Management*, *2*(2), pp.157-167.

[11] Indro and and Lee, Biases in Arithmetic and Geometric Averages as Estimates of Long-Run Expected Returns and Risk Premia, *Financial Management*, Vol. 26, No. 4 (Winter, 1997), pp. 81-90.

[12] Elsner, S. and Krumholz, H.C., 2013. Corporate valuation using imprecise cost of capital. *Journal of Business Economics*, *83*(9), pp.985-1014.

[13] Blume ME (1974) Unbiased estimators of long-run expected rates of return. *J Am Stat Assoc* 69:634–638. https:// doi. org/ 10. 1080/ 01621 459. 1974. 10480 180

[14] Competition Economists Group – *Estimating MRP for the ERA 2022 RoRI* – August 2022, paragraph 18, page 7.

[15] Indro and and Lee, Biases in Arithmetic and Geometric Averages as Estimates of Long-Run Expected Returns and Risk Premia, *Financial Management*, Vol. 26, No. 4 (Winter, 1997), pp. 81-90.

of monthly returns are used as a proxy for determining the future expected N-month market return."

17. Hence, as invariably any proposed estimator may be considered as a proxy, then useful criteria for adjudging different constructions of estimators, in this case of the geometric mean, may be the statistical properties of those estimators.

18. Dimson, Marsh and Staunton provide precedence as to one way to calculate the market risk premium. Economic arguments aside, there is at present little statistical support in the literature as to whether this common method of estimating the market risk premium is superior to that of Kaserer and the Authority. Kaserer instead set out a well-designed simulation study to explore in depth the properties of the different estimators applied to the excess market returns, and evaluate these different estimators in terms of bias and variance across a number of different data scenarios, including the presence of serial autocorrelation, mean reversion, and heteroskedasticity within a time series.

19. Algebraically, neither of the two calculations uniformly give results higher than the other. CEG give an example where their method (difference ***in*** means) is greater than the ERA method (mean ***of*** differences). A simple counter example shows this is not always the case:

    - risk free returns of 1% and 3%. Market returns of 3% and 7%.
    - the difference in means is 2.986% and the mean of the difference is 2.995%.

20. In practice we would expect the difference in means calculation to be higher. For the simulations, over all 10,000 replications over many simulation parameter settings, the difference in means approach always produced a larger geometric mean. The difference ranged from 4bps larger to 27bps with a mean of 12bps. The differences increase with increasing volatility (ω in simulation model) and with increasing heteroskedasticity (α in simulation model). The results show immaterial difference between the two geometric mean approaches.

21. An attractive property of the mean of differences approach is that if the excess returns are constant over years, this approach results in a geometric mean that is that constant return, whereas the difference in means will not do this in general (only if the risk-free rate is also constant over time).

2.1.2 Weighting the geometric mean

22. CEG initially contend that the Authority should assign 100% weight to the arithmetic mean and 0% weight to the geometric mean. They draw on recommendations from respected finance textbooks to apply the arithmetic mean.[16]

23. The use of the arithmetic mean as an unbiased estimator of the market risk premium is correct if the investment horizon is a single period horizon.

24. However, CEG relax the initial contention within the same paragraph and state that: *"If, nonetheless, the ERA, in our view incorrectly, determined that its task was to estimate the IRR on an "invest and accumulate" investment strategy then the ERA should use the weighting scheme from the literature (e.g., as estimated by Indro and Lee (1997))."*

[16] Competition Economists Group – *Estimating MRP for the ERA 2022 RoRI* – August 2022. Paragraph 18, page 7.

25. There is strong evidence in the literature supporting an Indro and Lee type scheme that assigns partial weight to the geometric mean depending on the length of the time series being estimated (T) and the length of the time horizon (N) to which the estimate of long-term returns, and hence the market risk premium, is to be applied to.[17] The weight towards the geometric mean increases with horizon N.

26. Kaserer reports nine different schemes to assigning weights between the arithmetic and geometric means to provide a statistically optimal estimator of the discount factor from a series of returns.[18] The resulting estimators are applied directly to excess returns so as to estimate the market risk premium.

27. The illustrative example given from Brealey, Myers and Allen[19] is useful in understanding some of the behaviour of arithmetic and geometric means. However, in assuming a known probability mass function for the returns, it is equivalent to an infinite training span of data, which in both the Indro and Lee and Jacquier et al. methods revert to the arithmetic mean. The methods in Indro and Lee and Jacquier et al. account for uncertainty in the estimation of underlying returns as well as the horizon over which the returns are forecast.

28. For a horizon of one period, the Indro and Lee method reverts to the arithmetic mean, regardless of the estimation span. Jacquier et al. produces a return estimate lower than the arithmetic mean under both the unbiased and minimum MSE options. The Brealey et al. (2006) text book that CEG cites in turn cites Jacquier, Kane, and Marcus (2005) to note that "When future returns are forecasted to distant horizons, the historical arithmetic means are upward-biased. This bias would be small in most corporate-finance applications, however."

29. The Berk and DeMarzo (2020)[20] text book that CEG cites gives an illustrative example of returns from a known probability mass function (*Arithmetic Average Returns Versus Compound Annual Returns*, p. 368). In this example the authors state that the assumption to "view past returns as independent draws from the same distribution" is needed for the arithmetic mean to be an unbiased estimator of the expected return.

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[17] Blume ME (1974) Unbiased estimators of long-run expected rates of return. J Am Stat Assoc 69:634–638. https:// doi. org/ 10. 1080/ 01621 459. 1974. 10480 180.

Cooper, I., 1996. Arithmetic versus geometric mean estimators: Setting discount rates for capital budgeting. *European Financial Management*, *2*(2), pp.157-167.

Indro and and Lee, Biases in Arithmetic and Geometric Averages as Estimates of Long-Run Expected Returns and Risk Premia, Financial Management, Vol. 26, No. 4 (Winter, 1997), pp. 81-90.

Jacquier, E., Kane, A. and Marcus, A.J., 2003. Geometric or arithmetic mean: A reconsideration. *Financial Analysts Journal*, *59*(6), pp.46-53.

Jacquier, E., Kane, A. and Marcus, A.J., 2005. Optimal estimation of the risk premium for the long run and asset allocation: A case of compounded estimation risk. *Journal of Financial Econometrics*, *3*(1), pp.37-55.

[18] Kaserer, Christoph. "Estimating the market risk premium for valuations: arithmetic or geometric mean or something in between?" *Journal of Business Economics* 92, no. 8 (2022): 1373-1415.

[19] Brealey, R.A., Myers, S.C., Allen, F. and Krishnan, V.S., 2006. Corporate finance (Vol. 8). Boston et al.: McGraw-Hill/Irwin.

[20] Berk, J. and DeMarzo, P., 2020. Corporate Finance: Global Edition (5. utg.).

30. CEG state without providing supporting references that:

> *"Even with autocorrelation the arithmetic mean remains the best estimate of the unconditional expected return. Negative autocorrelation simply means that the conditional estimate will be higher/lower if the previous period's return was lower/higher than average. Negative autocorrelation does not mean that the arithmetic average will overstate the unconditional expected return."*

31. There are numerous authors that argue that negative serial autocorrelation leads to the arithmetic average overstating the unconditional expected long-term return. Kaserer demonstrates that an equal weighting of the geometric and arithmetic mean estimates outperforms the arithmetic mean in the presence of strong serial autocorrelation. Other authors provide similar results.[21]

32. A stronger argument to retain the arithmetic mean alone, based on serial autocorrelation, is to claim that serial autocorrelation is either weak or non-existent. Negative serial autocorrelation is a known phenomenon that occurs in numerous sources of market data world-wide.[22] However, serial autocorrelation is not present in every data source, and its expression in excess returns may be less pronounced than in market return data from which those excess returns are derived. Jacquier et al. state that

> *"Strong cases are made in recent studies that the estimate of the market risk premium should be revised downward. Our result compounds this by stating that even these lower estimates of mean return should be adjusted further downward when used to predict long-term returns."*

Hence, even if serial autocorrelation is weak there is still justification for not relying solely on the arithmetic mean estimator in scenarios where the investment horizon is long-term.

33. Regarding correlated returns, consider the toy example taken from Brealey, Myers and Allen. Let X be a discrete random variable with probability mass function (p.m.f):

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[21] Kaserer, Christoph. "Estimating the market risk premium for valuations: arithmetic or geometric mean or something in between?" Journal of Business Economics 92, no. 8 (2022): 1373-1415.

Cooper, I., 1996. Arithmetic versus geometric mean estimators: Setting discount rates for capital budgeting. European Financial Management, 2(2), pp.157-167.

Indro and and Lee, Biases in Arithmetic and Geometric Averages as Estimates of Long-Run Expected Returns and Risk Premia, Financial Management, Vol. 26, No. 4 (Winter, 1997), pp. 81-90.

[22] For example:
Shiller RJ (2014) Speculative asset prices. *Am Econ Rev* 104(6):1486–1517.

Spierdijk L, Bikker JA, van den Hoek P (2012) Mean reversion in international stock markets: an empirical analysis of the 20th Century. *J Int Money Financ* 31(2):228–249

Pástor, Ľ. and Stambaugh, R.F., 2012. Are stocks really less volatile in the long run?. *The Journal of Finance*, 67(2), pp.431-478.

Campbell JY (2003) Consumption-based asset pricing. In: Constantinides GM, Harris M, Stulz RM (eds)*Handbook of the economics of finance*. Elsevier, Amsterdam, pp 801–885

Poterba JM, Summers LH (1988) Mean reversion in stock prices: evidence and implications. *J Financ*

Econ 22(1):27–59.

$$P(X=x)=\begin{cases}\frac{1}{3}, & \text{if } x=0.9\\ \frac{1}{3}, & \text{if } x=1.1\\ \frac{1}{3}, & \text{if } x=1.3\\ 0, & \text{otherwise}\end{cases}$$

Consider a returns series for year t where:

$$\begin{aligned} R_1 &= X_1, \text{ and}\\ R_{t+1} &= X_{t+1}+\gamma(\mu-R_t), \text{ for } t=1,2,\ldots \end{aligned}$$

and X_t are independent realisations from the random variable X and $\gamma \in (-1,1)$.

From the p.m.f. of X:

$$\mu = E[X] = 0.9\times\left(\frac{1}{3}\right)+1.1\times\left(\frac{1}{3}\right)+1.3\times\left(\frac{1}{3}\right) = 1.1 \text{ , and}$$

$$E[X^2] = 0.9^2\times\left(\frac{1}{3}\right)+1.1^2\times\left(\frac{1}{3}\right)+1.3^2\times\left(\frac{1}{3}\right) = 1.23\dot{6}.$$

Also, the expected return at each year individually is the same as for the uncorrelated example, i.e. $E[R_t]=E[X]=\mu,\ \forall t$.

We can show that the expected return over a 2-year period is

$$\begin{aligned} E[R_1R_2] &= E\big[X_1\big(X_2+\gamma(\mu-X_1)\big)\big]\\ &= \mu^2-\gamma(E[X^2]-\mu^2)\\ &= \mu^2-\gamma\sigma_X^2 \end{aligned}$$

For, say, $\gamma=0.6$ this gives an expected 2-year return of 1.194. Again, this estimate is based on perfect knowledge of the p.m.f. generating the returns, rather than parameters estimated from a finite span of data. How is it that the opportunity cost of capital for this investment can be the same as for the uncorrelated example in Brealey, Myers and Allen that had an expected 2-year return of $1.1^2=1.21$?

34. The extreme example with

$$P(Return=x)=\begin{cases}\frac{1}{2}, & \text{if } x=3\\ \frac{1}{2}, & \text{if } x=0\\ 0, & \text{otherwise}\end{cases}$$

is an example where there is an absorbing boundary of zero for an accumulation investment strategy. The geometric mean will be zero in this example “provided our time series has at least one year with a -100% return event – which is very likely given the annual probability of 50%).”. This is correct, although if the market has had a period of -100% return there would be nothing left to invest in, unless a subsequent return was infinite. The expected return in any given year would be 50% (not 100%) and if this market could be played like a casino game, then this return could be realised on average for the stake placed at each period. We do not consider this singular example very useful for illustrating the estimation of historic MRP.

2.1.3 Investment horizon

35. In a footnote Brealey, Myers and Allen[23] state:

> *"You sometimes hear that the arithmetic average correctly measures the opportunity cost of capital for one-year cash flows, but not for more distant ones. Let us check. Suppose that you expect to receive a cash flow of $121 in year 2. We know that one year hence investors will value that cash flow by discounting at 10% (the arithmetic average of possible returns). In other words, at the end of the year they will be willing to pay PV1 = 121/1.10 = $110 for the expected cash flow. But we already know how to value an asset that pays off $110 in year 1—just discount at the 10% opportunity cost of capital. Thus PV0 = PV1/1.10 = 110/1.1 = $100. Our example demonstrates that the arithmetic average (10% in our example) provides a correct measure of the opportunity cost of capital regardless of the timing of the cash flow."*

To follow this example from Brealey, Myers and Allen, we have shown above that including $\gamma = 0.6$ results in a cash flow of $119.40 in year 2, while keeping the expected return at each year to 10%. Serial correlation is irrelevant if we only look at one period's return but it cannot be ignored if looking beyond a horizon of one period.

36. We show in the simulation study in this report that mean squared error (whether it be in terms of error in annualised parameter estimate or cumulative return estimate) for the arithmetic mean increases sharply relative to other estimators as the horizon length increases beyond one period. This emphasises that the argument put forward by CEG regarding the absorbing boundary example is something of a 'straw man'. Their investment scenario is artificial in that the investor is not able to diversify away the risk of investing in a single asset and according to their level of aversion to risk. Moreover, the argument has nothing to do with the question at hand, namely identifying a weighting between the geometric mean and the arithmetic mean to provide the MRP estimator that minimises either overall bias or prediction error at a given horizon N. The straw man argument does not lead to a better statistical estimator of the market risk premium.

37. The statistical impact of the choice of horizon can be seen in the results of the simulation study below, as is supported by the theoretical results provided in the literature. However, the choice of which horizon is most appropriate for the ERA remains an economic question, rather than statistical.

38. CEG refer to an investment strategy of investing an equal amount each period and at the end of each period withdrawing gains to, or replenishing losses from, another reserve of capital. It is up to ERA to determine if such a strategy accurately reflects the nature of the investment in regulated assets. If it does, then a single-year horizon may be appropriate. The performance of such a strategy overall depends on the risk and returns from the reserve of capital also, and how these returns are correlated with the returns of the market. If CEG are proposing that the rate of return be set to match the returns of such a strategy, then the strategy should be considered as a whole. If the reserve of capital that takes gains and replenished losses has the same returns as the market then the multi-period return of such a strategy is the same as compounded returns.

39. Compared to our simulations the theoretical results are more conservative in the weight that they assign the geometric mean, and hence may be considered as a lower bound to that weighting.

[23] Brealey, R.A., Myers, S.C., Allen, F. and Krishnan, V.S., 2006. Corporate finance (Vol. 8). Boston et al.: McGraw-Hill/Irwin.

From a statistical perspective the estimator of Jacquier et al.[24] that minimises the MSE would be favoured over the Indro and Lee (1998) method, as Indro and Lee consider only bias in the MRP estimator and not overall prediction accuracy of future returns. The Jacquier, Kane, and Marcus estimator places more weight on the geometric mean than the Indro and Lee estimator.

2.2 Dr Martin Lally – Tests of Mean Stationarity for Australian Share Market Returns Data

40. Dr Lally provides tests of three different possible sources of non-stationarity:
    - a gradual drift downwards in the population mean for excess returns as investors have become more diversified and the cost of forming a well-diversified portfolio has fallen (i.e. the process may exhibit trend stationarity)
    - the population mean of the excess returns (i.e. the MRP) experiences occasional changes (i.e. the process is non-stationary due to regime shifts).
    - the process is autoregressive (outcomes are linearly related to past outcomes) with a unit root or explosive process. (i.e. the process may exhibit difference stationarity)
41. We accept Dr Lally's position that there is little evidence of a gradual drift in the population mean for excess returns. However, we do comment that the test of linear drift applied fits a very simplified linear model to the time series, with the model underpinning Dr Lally's test likely a misspecification of the data. This comment would argue that drift over a longer period of time is more likely to be a non-linear or fluctuating process rather than linear. This non-linear view of time series would be more concordant with the data-driven detection of regime shifts.
42. Dr Lally applies an ANOVA F-test separately to time series of the excess returns, real returns and nominal returns, based on either regular segmentation (iteratively into halves, thirds, quarters and fifths) of the time series, or a segmentation based on two major market events (1937 and 1988, the latter due to the introduction of dividend imputation). There is a couple of weaknesses with this approach. The first is that an F-test will likely underperform in terms of higher rates of type II error (i.e., falsely identifying no difference between different time periods when actual difference exists) simply because any serial autocorrelation (weak or otherwise) in the time series has not been taken into account by the test.
43. The second issue is the boundaries of each potential 'regime' identified by Dr Lally have been set ex-ante, either arbitrarily across a regular grid and subject to all the known weaknesses of systematic sampling of miss-specified interval location and interval width, or based on known structural breaks in the market. The issue with known structural breaks is that due to possible lagged effects the impact of the structural break may not occur coincidentally with the timing of the structural break itself.
44. Two data-driven methods may in future be trialled to test for regime shifts that are designed to overcome the limitations of the ANOVA tests applied by Dr Lally, namely identifying potential regime shifts ex-poste rather than ex-ante. A first test can be supported by a Markov switching

[24] Jacquier, E., Kane, A. and Marcus, A.J., 2005. Optimal estimation of the risk premium for the long run and asset allocation: A case of compounded estimation risk. *Journal of Financial Econometrics*, *3*(1), pp.37-55.

autoregressive model (or similar)[25], which treats the boundary points of each potential regime as hidden variables. Inference as to whether different regimes exist in the times series is then tested through maximum likelihood methods (either Bayesian or frequentist). A second test based on Sequential Regime Shift Detection (SRSD) methods[26] developed from iterative partitioning of the CUSUM chart of the time series. Both methods may be vectorized (i.e. consider multivariate time series simultaneously, as may occur if the excess returns, real returns and nominal returns were bundled into the one analysis).

45. Of these two data-driven methods the SRSD is perhaps more attractive insofar as it does not assume any serial autocorrelation structure in the data whatsoever, given Dr Lally's coherent arguments as to why serial autocorrelation should not exist within the time series of the excess returns, in discussion of the Augmented Dickey Fuller (ADF) test for unit roots.[27] However, under a data-driven paradigm if serial autocorrelation in a time series is detected then it is more an indicator that some observed, as yet unexplained process is occurring beyond the scope of theoretical justification. Hence, exploring a Markov switching process may remain useful.

46. In summary, the ANOVA F-test applied by Dr Lally can be deemed insufficient to detect ex-poste structural breaks in time series, as boundaries to qualitatively different regimes of time series behaviour. More sophisticated statistical tests apply. Until such methods are rigorously applied then from a precautionary principle one would not fully accept a null hypothesis of no difference in time series behaviour between different time periods.

47. Dr Lally observes that his results are *"consistent with visual examination of the 30-year rolling average of excess returns, as calculated by the AER, i.e., there is very little variation in the 30-year average around the overall mean excess return of 6.4%."*

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[25] For example:

Krolzig, HM. (1997). The Markov-Switching Vector Autoregressive Model. In: Markov-Switching Vector Autoregressions. Lecture Notes in Economics and Mathematical Systems, vol 454. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-51684-9_2.

Zeileis A., Shah A., Patnaik I. (2010), Testing, Monitoring, and Dating Structural Changes in Exchange Rate Regimes, *Computational Statistics and Data Analysis*, 54(6), 1696--1706. http://dx.doi.org/10.1016/j.csda.2009.12.005.

[26] For example:

Luca Stirnimann, Alessandra Conversi, Simone Marini, Detection of regime shifts in the environment: testing "STARS" using synthetic and observed time series, *ICES Journal of Marine Science*, Volume 76, Issue 7, December 2019, Pages 2286–2296, https://doi.org/10.1093/icesjms/fsz148.

Josua Gösmann. Christina Stoehr. Johannes Heiny. Holger Dette. "Sequential change point detection in high dimensional time series." Electron. J. Statist. 16 (1) 3608 - 3671, 2022. https://doi.org/10.1214/22-EJS2027.

[27] Dr Martin Lally – *Tests of Mean Stationarity for Australian Share Market Returns Data* -June 2022 p6.

"This would require an unprecedented degree of informational inefficiency in a market, involving expected excess returns (i.e., true MRPs) that would be negative whenever the preceding year's excess return was negative and true MRPs that were very large whenever excess returns in the previous year were very large. Such a situation would also be incompatible with any version of the CAPM, in which the true MRP is a reward for bearing risk. Thus, conducting the ADF test on excess returns would seem to be pointless."

48. If, following more rigorous testing, mean stationarity of the time series is assumed to be supported by the data then including earlier periods in the time series (say prior to 1988) appears to be irrelevant. Under mean stationarity the last 30 years of data would be sufficient as the last 140 years of data to acquire a reliable estimate of the overall mean excess return.

49. However, estimates of the market risk premium based on different time series lengths do not appear stationary, with more recent estimates (5.30% for 2000-2021) being qualitatively higher than estimates based on the longer time series (4.56% for 1958-2021).[28] The impact of such differences between estimates of the market risk premium based on different data supports will clearly be sizeable on the return on equity calculation. However, such differences (ranging over an interval of 0.74%) would not have been statistically significant given the methods applied by Dr Lally.

50. Consequently, Dr Lally's report on mean stationarity should have little bearing on the Authority's methods of estimating the market risk premium. Either:

   - The design of tests for stationarity between different segments of the time series were weak (i.e., suffered from overly high rates of Type II error by design), and risk a false negative assertion that excess returns are stationary when in truth they may not be. In this scenario one would place more weight on more recent time series when averaging estimates of the market risk premium.

   - Dr Lally's support of the null hypothesis of mean stationarity in excess returns holds true. In this scenario the argument may be made to discount in part earlier time series, by virtue that they add little information to the estimate of the market risk premium derived from more recent time periods.

51. The Authority's method of deriving the historical estimate of the market risk premium already provides more weight to recent time periods than earlier period. Consequently, there is little reason to alter the Authority's method given Dr Lally's findings.

[28] Competition Economists Group – *Estimating MRP for the ERA 2022 RoRI* – August 2022, Table 2-1, p4. Reported values based on current ERA method, with gamma set to 0.5.

3 Simulation Study

The following is a simulation study to understand which methods for estimating the historical market risk premium perform well under various conditions. Estimates from each method were evaluated against known, simulated returns from the data generating process.

3.1 Data

We followed the return generating simulation based on a GARCH(1,1) model used in Kaserer (2022)[29]. Kaserer defined the market risk premium r as

$$r = \ln(1 + r_m) - \ln\left(1 + r_f\right)$$

where r_m and r_f are market returns and risk-free rate respectively. We used the difference in rate, rather than the ratio of growth, as this is the definition of market risk premium used by ERA and CEG. Therefore, we have

$$r = r_m - r_f$$

This log return was simulated as follows:

$$\ln(r_t) = \mu + \gamma\left(\mu - \ln(r_{t-1})\right) + \sigma_t \varepsilon_t$$
$$\sigma_t^2 = \omega^2 + \alpha\sigma_{t-1}^2 + (\alpha - \beta)\sigma_{t-1}^2(\epsilon_t^2 - 1)$$

The desired market risk premium for the simulations was set to be 0.05 and 0.07. The mean of the log returns generating function was set to be $\mu = \ln(1 + r) - 0.5\omega^2$. Once the market risk premium was generated over the required span, the market returns were defined as this premium plus a constant risk-free rate of 0.04. It was necessary to produce simulated series for both the market and risk-free returns in order to evaluate the alternative method of geometric mean excess returns using the difference in geometric means of market returns and risk-free returns (i.e., the method proposed by CEG). The risk-free returns could have been simulated by a stochastic process, however a static return with the excess returns modelled by the GARCH process was deemed fit for the purpose of this study.

Other parameters in the simulation model are:

- Mean of log returns, μ
- Volatility $\omega \in \{0.15, 0.20\}$
- Heteroskedasticity $\alpha \in \{0.0, 0.6\}$. β is set at $\alpha/2$ as in Kaserer.
- Autocorrelation $\gamma \in \{-0.2, 0.0, 0.2, 0.5\}$. The last two settings give a weakly stationary mean-reverting process with a negative autocorrelation.
- ε_t and ϵ are both realisations from Standard Normal distribution

For each combination of settings 10,000 time series of annual returns were generated from the model.

[29] Kaserer, Christoph. 2022. "Estimating the Market Risk Premium for Valuations: Arithmetic or Geometric Mean or Something in Between?" *Journal of Business Economics* 92 (8): 1373–1415.

3.2 Estimation

We applied a variety of candidate estimation methods to the simulated series, creating an estimate of the MRP for each. There were four main estimators trialled, with several parameter settings for each:

- Weighted average of arithmetic and geometric means

$$\widehat{\mathrm{MRP}}^{(W)} = w\mathrm{Amean} + (1-w)\mathrm{Gmean}$$

Where:

weighting of arithmetic mean is by $w \in [0,1]$

$$\text{Amean} = \frac{1}{T}\sum_{t=1}^{T}(r_m - r_f)$$

diff of gmean: $\text{Gmean} = \exp\left(\frac{1}{T}\sum_{t=1}^{T} ln(1+r_m)\right) - \exp\left(\frac{1}{T}\sum_{t=1}^{T} ln(1+r_m)\right)$

gmean of diff: $\text{Gmean} = \exp\left(\frac{1}{T}\sum_{t=1}^{T} ln\left(1+r_m-r_f\right)\right) - 1$

- Weighted average with weight as specified in Indro and Lee[30]. Note that Indro and Lee and Blume[31] define their weighting on the arithmetic and geometric means of the "N-period relative" which is the expected return over the horizon, rather than as the weighted mean of the annualised estimates.

$$\widehat{\mathrm{MRP}}^{(IL)} = (w_{\mathrm{IL}}(1+\mathrm{Amean})^N + (1-w_{\mathrm{IL}})(1+\mathrm{Gmean})^N)^{1/N}$$

Where:

$$w_{\mathrm{IL}} = \frac{(T-N)}{(T-1)}$$

- Jacquier, Kane and Marcus[32]. Annualised cumulative return estimated by:

$$\widehat{\mathrm{MRP}}^{(JKM)} = \exp\left(\hat{\mu} + \frac{1}{2}k\sigma^2\right)$$

Options for this method are:

unbiased: $k = 1 - N/T$

minMSE : $k = 1 - 3N/T$

Sample estimates were calculated as follows:

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[30] Indro, Daniel C, and Wayne Y Lee. 1997. "Biases in Arithmetic and Geometric Averages as Estimates of Long-Run Expected Returns and Risk Premia." *Financial Management*, 81–90.

[31] Blume ME (1974) Unbiased estimators of long-run expected rates of return. J Am Stat Assoc 69:634–638. https://doi.org/10.1080/01621459.1974.10480180.

[32] Jacquier, Eric, Alex Kane, and Alan J Marcus. 2005. "Optimal Estimation of the Risk Premium for the Long Run and Asset Allocation: A Case of Compounded Estimation Risk." *Journal of Financial Econometrics* 3 (1): 37–55.

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} \ln\left(1 + m_{m,t} - r_{f,t}\right)$$

$$\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(\ln\left(1 + m_{m,t} - r_{f,t}\right) - \hat{\mu}\right)^2$$

- Generalized (power) mean – with various powers p.

  diff of gmeans:

$$\widehat{\mathrm{MRP}}^{(G)} = \left(\frac{1}{T}\sum_{t=1}^{T}(1 + r_m)^p\right)^{1/p} - \left(\frac{1}{T}\sum_{t=1}^{T}\left(1 + r_f\right)^p\right)^{1/p}$$

  gmean of diff:

$$\widehat{\mathrm{MRP}}^{(G)} = \left(\frac{1}{T}\sum_{t=1}^{T}\left(1 + r_m - r_f\right)^p\right)^{1/p} - 1$$

  For $p = 1$ this is equal to the arithmetic mean and as $p \to 0$ this tends to the geometric mean.

For each method there is also the choice to be made of the span of data to use for estimation and the span of the horizon over which prediction is needed. In this study, each method was applied to simulated series with the following lengths:

- span of data for estimation, $T = 64{,}100{,}138$. These spans match the span to present from 1958 and 1883 and an arbitrary 100 years at times used in the literature.
- span of data for horizon, $N = 1{,}2{,}3{,}4{,}5{,}10$

3.3 Evaluation

We scored the performance of each estimation method, in terms of estimation error relative to the known MRP of the simulated series over the horizon of interest.

The mean squared error (MSE) is reported, as well as its components, the estimated bias and standard error of the estimate. The MSE can be decomposed into the sum of the squared bias and the variance of the estimate as follows:

$$\begin{aligned} MSE(\hat{r}) &= E[(\hat{r} - r)^2] \\ &= (E[\hat{r} - r])^2 + E\left[\left(\hat{r} - E(\hat{r})\right)^2\right] \\ &= \mathrm{bias}(\hat{r})^2 + \mathrm{Var}(\hat{r}) \end{aligned}$$

There are more than one possible way to calculate mean squared error. Each estimation method produces, for defined method parameters such as weighting scheme and also for defined horizon N and estimation data spans T, an estimate of the annual excess return $\hat{r}$. These estimates were compared to the observed returns from the simulations where $r_{it,m}$ and $r_{it,f}$ are the simulated market return and risk-free returns respectively at time t for simulation $k = 1, \ldots, K$ with K in this case 10,000.

The forms of the MSE used were:

1. MSE of the annualised observed growth in the market risk premium and its estimate.

$$\text{MSE}(\hat{r}) = \frac{1}{K}\sum_{k=1}^{K}\left(\left(\prod_{t=T+1}^{T+N}\left(1 + r_{kt,m} - r_{kt,f}\right)\right)^{1/N} - (1+\hat{r})\right)^2$$

2. MSE of the observed cumulative returns over the horizon and its estimate.

$$\text{MSE}(\hat{r}) = \frac{1}{K}\sum_{k=1}^{K}\left(\left(\prod_{t=T+1}^{T+N}\left(1 + r_{kt,m} - r_{kt,f}\right)\right) - (1+\hat{r})^N\right)^2$$

The difference between the two methods is whether the squared error is measured in terms of the annualised MRP estimate or in terms of the cumulative return at horizon N that the MRP implies. The method in Jacquier et al. uses the second definition of MSE.

Below is a numeric example to illustrate the difference between the two MSE measures, based on hypothetical ‘observed’ returns over a horizon of 3 years and an estimate of 7.05%.

Table 1 Numerical example of measures of MSE

	t = 1	t=2	t= 3	Compounded
Estimate $\hat{r}$	7.05%	7.05%	7.05%	22.68%
Observed r	6%	7%	8%	22.49%
Observed r_m	10%	11%	12%	36.75%
Observed r_f	4%	4%	4%	12.49%

The first measure of MSE is that of annualised estimate of the excess returns

$$
\begin{aligned}
MSE(\hat{r}^1) \quad &= \frac{1}{K}\sum_{k=1}^{K}\left(\left(\prod_{t=T+1}^{T+N}\left(1 + r_{kt,m} - r_{kt,f}\right)\right)^{\frac{1}{N}} - (1+\hat{r})\right)^2 \\
&= \left((1.06 \times 1.07 \times 1.08)^{\frac{1}{3}} - 1.0705\right)^2 \\
&= \left(1.2249^{\frac{1}{3}} - 1.0705\right)^2 \\
&= (1.06997 - 1.0705)^2 \\
&= 2.82 \times 10^{-7}
\end{aligned}
$$

This gives a different result to the MSE below based on the cumulative returns at the horizon.

$$
\begin{aligned}
MSE(\hat{r}^N) \quad &= \frac{1}{K}\sum_{k=1}^{K}\left(\left(\prod_{t=T+1}^{T+N}\left(1 + r_{kt,m} - r_{kt,f}\right)\right) - (1+\hat{r})^N\right)^2 \\
&= \left((1.06 \times 1.07 \times 1.08) - 1.0705^3\right)^2 \\
&= (1.2249 - 1.0705^3)^2 \\
&= (1.2249 - 1.2268)^2 \\
&= 3.33 \times 10^{-6}
\end{aligned}
$$

These MSE measures give different results in the sense that the best method with respect to one MSE definition will not necessarily be the best with respect to the other. This is due to the fact that the MSE definitions are measuring different types of errors that are related but not equivalent.

3.4 Results

The results of the study show that overall, the weighted average of the arithmetic and geometric means is a good estimator of the market risk premium over a range of scenarios. As expected from the theoretical results in the literature, the arithmetic mean should receive more weight when the horizon of interest is a single period. As the horizon is extended, more weight should be placed on the geometric mean in order to keep the MSE low.

The results for specific simulation settings given in Table 4 through to Table 15 show that the geometric mean should be given a non-zero weight even when there is no autocorrelation ($\gamma = 0$). Moreover, as the time horizon N increases, the optimal weighting between arithmetic and geometric means, in terms of minimising MSE, converges for different autocorrelation scenarios. This implies autocorrelation is less of a confounding issue in the weighting of the MRP estimator when time horizons are large.

The preferred method depends on the loss function defined by the MSE that is most relevant to the estimation task. When the MSE is based on the parameter estimate of the annualised market risk premium, there is less weight placed on the arithmetic mean than when the MSE is based on the estimated compounded return over the horizon of interest. As expected, when the MSE is based on the estimated compounded return over the horizon, the minimum MSE estimator of Jacquier et al. performs well.

3.4.1 Generalized mean compared to weighted average of means

The generalised mean with power parameter *p* gives results very similar to the weighted average of the arithmetic and geometric means with the arithmetic given the same weight *p*. Over all the combinations of simulation parameters and estimation parameters, the estimates for generalised mean and weighted mean did not differ by more than 0.00094. The two estimators can therefore be used interchangeably and we remove the generalised mean results from the analysis below.

3.4.2 Overall Performance

Table 2 and Table 3 shows which method performs best in a MSE sense over all simulations. This is shown for:

- the horizon of interest *horizon*;
- the length of data used in estimation (*est span* or T in the formulae);
- the estimation method (*est method*);
- the parameter value if any used for the estimation method (*est method param* which are defined along with the method definitions above, e.g. in the case of weighted mean method, a parameter of 0.2 means the weight of 0.2 is placed on the arithmetic mean);
- the method used for calculation of geometric mean excess returns (*geom mean method* takes values diff_of_gmean and gmean_of_diff as defined above); and
- the MSE of the annualised parameter estimate, where the mean is taken over all simulations over all simulation settings.

Table 2 shows results under the MSE with respect to the annualised estimate and Table 3 shows results for the MSE with respect to the cumulative return. As well as the best performing method for each combination of horizon, estimation span, methods with a mean MSE within 1% of the best are also shown. As expected from the theory, as the horizon of interest increases, more weighting should be placed on the geometric mean or excess returns in order to minimise MSE.

Table 2 Estimation settings with overall best MSE (for annualised estimate) for different spans.

horizon	est span	estimation method	estimation method param	geometric mean method	MSE	bias	SE
1	63	JKM	minMSE	——	0.00103	0.0013	0.03
1	63	JKM	unbiased	——	0.00104	0.00025	0.03
1	100	JKM	arithmetic	——	0.000663	-0.00011	0.024
1	100	JKM	minMSE	——	0.000659	0.00087	0.024
1	100	JKM	unbiased	——	0.000662	0.00022	0.024
1	138	JKM	arithmetic	——	0.000483	-1.3e-06	0.02
1	138	JKM	minMSE	——	0.000481	7e-04	0.02
1	138	JKM	unbiased	——	0.000482	0.00023	0.02
2	63	weighted_mean	0.3	diff_of_gmean	0.000925	0.0024	0.028
2	63	weighted_mean	0.3	gmean_of_diff	0.000929	0.0032	0.028
2	100	weighted_mean	0.3	diff_of_gmean	0.000592	0.0025	0.022
2	100	weighted_mean	0.3	gmean_of_diff	0.000597	0.0033	0.022
2	100	weighted_mean	0.4	gmean_of_diff	0.000597	0.00012	0.023
2	138	weighted_mean	0.3	diff_of_gmean	0.000435	0.0025	0.019
2	138	weighted_mean	0.4	diff_of_gmean	0.000438	-0.00054	0.019
2	138	weighted_mean	0.4	gmean_of_diff	0.000436	0.00017	0.019
3	63	weighted_mean	0.1	diff_of_gmean	0.000901	0.0042	0.028
3	63	weighted_mean	0.2	diff_of_gmean	0.000895	0.0012	0.028
3	63	weighted_mean	0.2	gmean_of_diff	0.000898	0.0021	0.028
3	100	weighted_mean	0.2	diff_of_gmean	0.000569	0.0013	0.022
3	100	weighted_mean	0.2	gmean_of_diff	0.000572	0.0022	0.022
3	138	weighted_mean	0.2	diff_of_gmean	0.000415	0.0013	0.019
3	138	weighted_mean	0.2	gmean_of_diff	0.000418	0.0023	0.019
4	63	weighted_mean	0.1	diff_of_gmean	0.000883	0.0021	0.028
4	63	weighted_mean	0.1	gmean_of_diff	0.000889	0.0031	0.028
4	63	weighted_mean	0.2	gmean_of_diff	0.000891	-4e-05	0.028
4	100	weighted_mean	0.1	diff_of_gmean	0.000562	0.0021	0.022
4	100	weighted_mean	0.2	diff_of_gmean	0.000566	-9.3e-04	0.022
4	100	weighted_mean	0.2	gmean_of_diff	0.000565	1.2e-05	0.022
4	138	weighted_mean	0.1	diff_of_gmean	0.000411	0.0022	0.019
4	138	weighted_mean	0.2	diff_of_gmean	0.000412	-8.6e-04	0.019
4	138	weighted_mean	0.2	gmean_of_diff	0.000411	8.1e-05	0.019
5	63	weighted_mean	0	diff_of_gmean	0.000885	0.0039	0.028
5	63	weighted_mean	0.1	diff_of_gmean	0.000877	0.00087	0.028
5	63	weighted_mean	0.1	gmean_of_diff	0.000880	0.00192	0.028
5	100	weighted_mean	0.1	diff_of_gmean	0.000556	0.00093	0.022
5	100	weighted_mean	0.1	gmean_of_diff	0.000560	0.00199	0.022
5	138	weighted_mean	0.1	diff_of_gmean	0.000405	0.0010	0.019
5	138	weighted_mean	0.1	gmean_of_diff	0.000409	0.0021	0.019
10	63	weighted_mean	0	diff_of_gmean	0.000861	0.0012	0.028
10	63	weighted_mean	0	gmean_of_diff	0.000867	0.0024	0.028
10	100	weighted_mean	0	diff_of_gmean	0.000544	0.0013	0.022
10	100	weighted_mean	0.1	gmean_of_diff	0.00055	-0.00073	0.022
10	138	weighted_mean	0	diff_of_gmean	0.000395	0.0013	0.019
10	138	weighted_mean	0.1	gmean_of_diff	0.000398	-0.00065	0.019

Note that these results are based on the mean MSE across all simulations and type of simulations. Different estimation methods and settings may be preferable for returns series with different characteristics. However overall, there is minimal difference in reported mean MSE values between the difference of geometric means compared to the geometric mean of the differences. Full tables are given in the Appendix with the best methods for each set of simulations.

Table 3 Estimation settings with overall best MSE (for cumulative returns) for different spans.

horizon	est span	estimation method	estimation method param	geometric mean method	MSE	bias	SE
1	63	JKM	minMSE	——	0.00103	0.0013	0.03
1	63	JKM	unbiased	——	0.00104	0.00025	0.03
1	100	JKM	arithmetic	——	0.000663	-0.00011	0.024
1	100	JKM	minMSE	——	0.000659	0.00087	0.024
1	100	JKM	unbiased	——	0.000662	0.00022	0.024
1	138	JKM	arithmetic	——	0.000483	-1.3e-06	0.02
1	138	JKM	minMSE	——	0.000481	7e-04	0.02
1	138	JKM	unbiased	——	0.000482	0.00023	0.02
2	63	weighted_mean	0.2	diff_of_gmean	0.00407	0.0055	0.028
2	63	weighted_mean	0.3	diff_of_gmean	0.00407	0.0024	0.028
2	63	weighted_mean	0.3	gmean_of_diff	0.00407	0.0032	0.028
2	100	weighted_mean	0.3	diff_of_gmean	0.00259	0.0025	0.022
2	100	weighted_mean	0.3	gmean_of_diff	0.00261	0.0033	0.022
2	138	weighted_mean	0.3	diff_of_gmean	0.00190	0.0025	0.019
2	138	weighted_mean	0.3	gmean_of_diff	0.00192	0.0033	0.019
3	63	weighted_mean	0.1	diff_of_gmean	0.00944	0.0042	0.028
3	63	weighted_mean	0.1	gmean_of_diff	0.00949	0.0053	0.028
3	100	weighted_mean	0.1	diff_of_gmean	0.00605	0.0043	0.022
3	100	weighted_mean	0.2	diff_of_gmean	0.00607	0.0013	0.022
3	100	weighted_mean	0.2	gmean_of_diff	0.00606	0.0022	0.022
3	138	weighted_mean	0.2	diff_of_gmean	0.00441	0.0013	0.019
3	138	weighted_mean	0.2	gmean_of_diff	0.00441	0.0023	0.019
4	63	weighted_mean	0	diff_of_gmean	0.0177	0.0051	0.028
4	63	weighted_mean	0	gmean_of_diff	0.0179	0.0063	0.028
4	100	weighted_mean	0	diff_of_gmean	0.0114	0.0052	0.022
4	100	weighted_mean	0.1	diff_of_gmean	0.0113	0.0021	0.022
4	100	weighted_mean	0.1	gmean_of_diff	0.0113	0.0032	0.022
4	138	weighted_mean	0.1	diff_of_gmean	0.00823	0.0022	0.019
4	138	weighted_mean	0.1	gmean_of_diff	0.00829	0.0033	0.019
5	63	weighted_mean	0	diff_of_gmean	0.0296	0.0039	0.028
5	63	weighted_mean	0	gmean_of_diff	0.0296	0.0051	0.028
5	100	weighted_mean	0	diff_of_gmean	0.0187	0.0040	0.022
5	100	weighted_mean	0	gmean_of_diff	0.0189	0.0052	0.022
5	138	weighted_mean	0	diff_of_gmean	0.0137	0.0041	0.019
5	138	weighted_mean	0.1	diff_of_gmean	0.0138	0.0010	0.019
5	138	weighted_mean	0.1	gmean_of_diff	0.0137	0.0021	0.019
10	63	weighted_mean	0	gmean_of_diff	0.184	0.0024	0.028
10	100	weighted_mean	0	gmean_of_diff	0.108	0.0024	0.022
10	138	weighted_mean	0	gmean_of_diff	0.076	0.0025	0.019

Figure 1 and Figure 2 show the performance of the estimator that is the weighted average of the geometric and arithmetic means. Each curve is the MSE over 10,000 simulated series for a particular setting of the simulated series. The MSE has been scaled by subtracting the minimum MSE achieved for each setting and dividing by the standard deviation, therefore each curve has a minimum at zero. This scaling is done for this plot to allow the minima to be seen more easily.

Each combination of settings, defined by choices of γ, α and ω are shown in a different colour. The length of the training span T varies across the grid columns from 63 to 138. The length of the horizon, N varies from 1 to 10 down the rows of the grid. The figure shows that for a one-year-ahead horizon, the arithmetic mean is preferred. As the horizon extends beyond one year, increasingly more weight should be given to the geometric mean in order to reduce mean squared error. This holds even for the case of *i.i.d.* log returns, when $\gamma = \alpha = 0$.

The differences between Figure 1 and Figure 2 show the importance of the definition of the MSE to the choice of weighting between arithmetic and geometric means. When the cumulative return over the horizon of interest is the estimate used in the MSE, the shift towards the geometric mean for longer horizons is slightly quicker than when the annualised market risk premium is the parameter in the MSE.

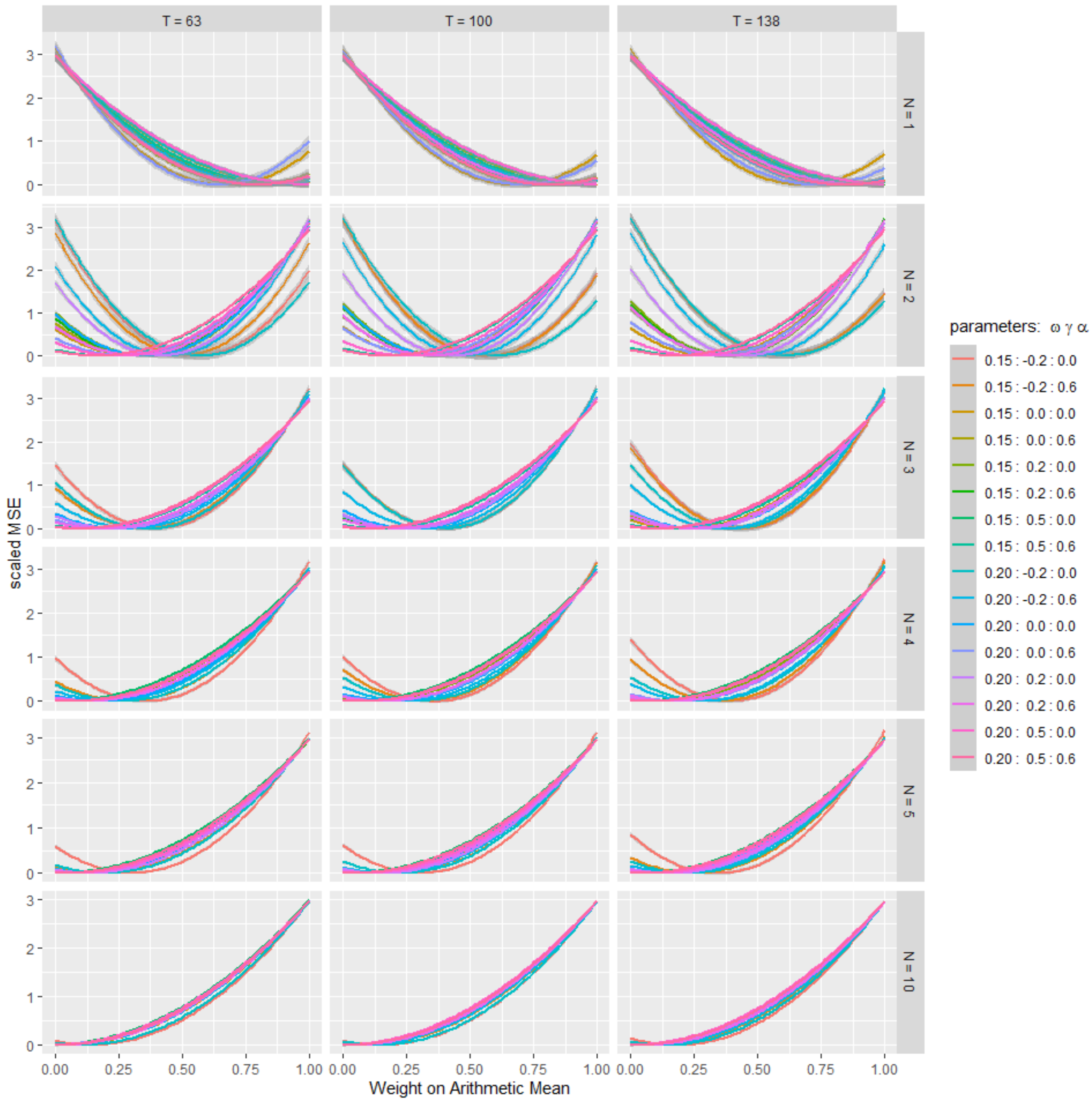

Figure 1 Scaled MSE for simulation settings and weighted average of arithmetic and geometric means. MSE is based on parameter estimate of annualised market risk premium.

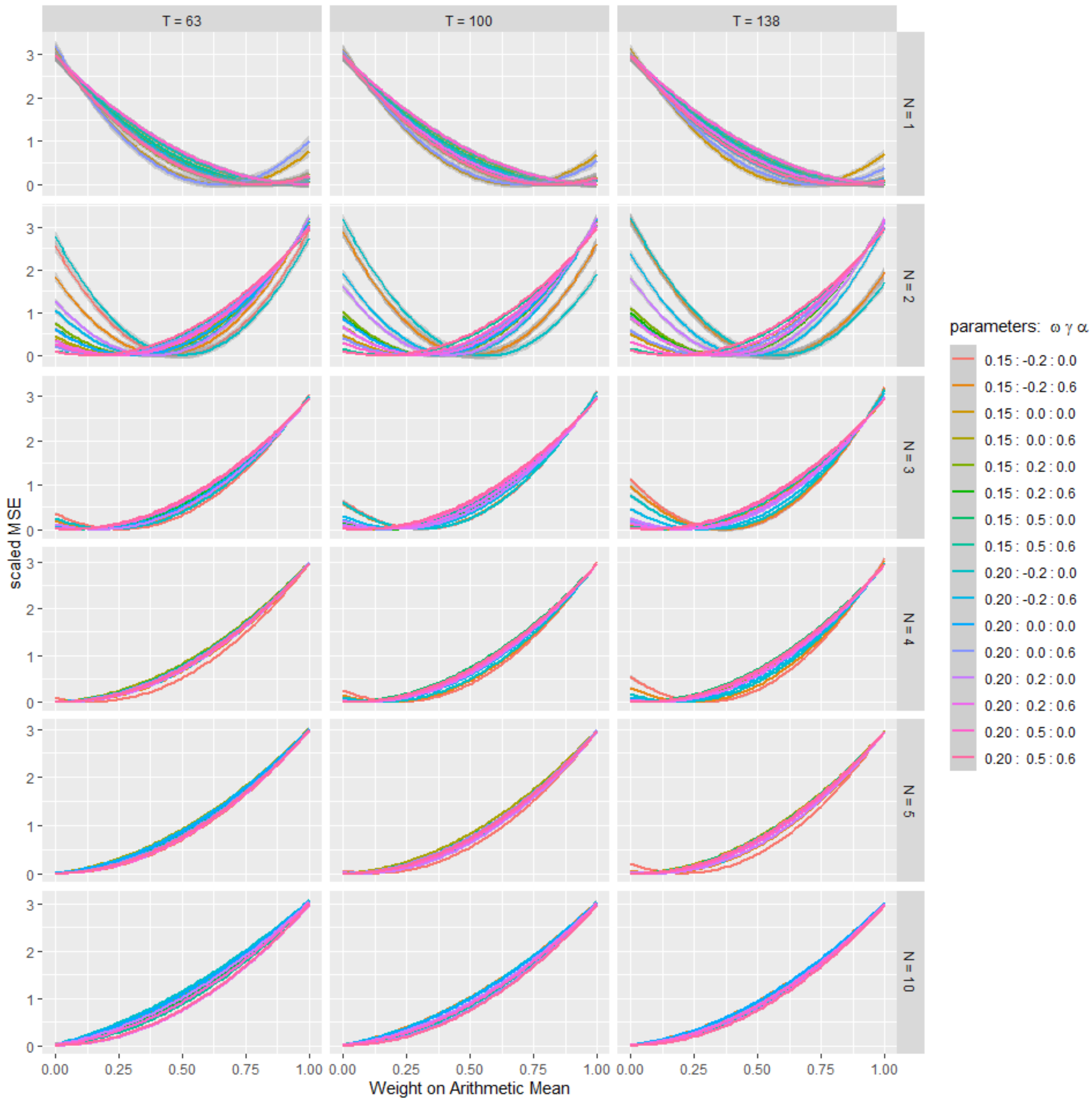

Figure 2 Scaled MSE for simulation settings and weighted average of arithmetic and geometric means. MSE is based on estimated cumulative return at the horizon.

4 Conclusion

The ERA needs to estimate the market risk premium from time series data of finite length with some degree of serial autocorrelation; for a horizon period of one or more years. The surveyed literature and simulation study in this report both support the use of a composite estimator of the market risk premium, that is a compromise between the arithmetic mean and geometric mean. While some finance text books advocate for the use of the arithmetic mean solely, these do not seem to account for sampling error in estimates or serial correlation in time series data.

We addressed the calculation of the geometric mean for excess returns. The current ERA method considers the series of excess returns and calculates the geometric mean of this series; the alternative method considers the market return series and risk-free rate series separately. We do not view either as being *wrong* statistically, although the current ERA method has the desirable property of preserving constant excess returns. While the current ERA method produces a smaller geometric mean, in practice we saw very little difference between the two in the simulation study. The choice of geometric mean method had little impact on what were the best weights to give the geometric mean and arithmetic mean to minimise mean squared error.

In the simulation study, we measured the performance of estimators in terms of the mean squared error. One can measure error in terms of cumulative excess returns at the horizon, or in terms of annualised estimates of the market risk premium. Although the exact form of mean squared error function used proved to be important to the choice of weighting, under either choice the ideal estimator placed more weight on the geometric mean as the horizon length increased.

5 Appendix

5.1 Best methods for MSE with respect to annualised estimate

Table 4 Minimum MSE methods for horizon of 1.

est span	ω	γ	α	est method	est parameter	geom mean method	MSE
63	0.2	0.5	0.6	JKM	minMSE	—	1.4e-03
63	0.2	0.5	0	JKM	unbiased	—	3.7e-04
63	0.2	0.2	0.6	JKM	minMSE	—	1.5e-03
63	0.2	0.2	0	JKM	arithmetic	—	5.1e-04
63	0.2	0	0.6	weighted_mean	0.6	diff_of_gmean	2.1e-03
63	0.2	0	0	weighted_mean	0.8	diff_of_gmean	7.2e-04
63	0.2	-0.2	0.6	JKM	minMSE	—	3.1e-03
63	0.2	-0.2	0	JKM	minMSE	—	1.1e-03
63	0.15	0.5	0.6	JKM	minMSE	—	6.2e-04
63	0.15	0.5	0	weighted_mean	0.8	diff_of_gmean	1.9e-04
63	0.15	0.2	0.6	JKM	minMSE	—	7.8e-04
63	0.15	0.2	0	JKM	minMSE	—	2.9e-04
63	0.15	0	0.6	weighted_mean	0.8	diff_of_gmean	1.1e-03
63	0.15	0	0	weighted_mean	0.7	diff_of_gmean	3.9e-04
63	0.15	-0.2	0.6	JKM	minMSE	—	1.7e-03
63	0.15	-0.2	0	weighted_mean	0.8	diff_of_gmean	6.1e-04
138	0.2	0.5	0.6	JKM	minMSE	—	6.4e-04
138	0.2	0.5	0	JKM	arithmetic	—	1.7e-04
138	0.2	0.2	0.6	JKM	minMSE	—	6.9e-04
138	0.2	0.2	0	JKM	arithmetic	—	2.3e-04
138	0.2	0	0.6	weighted_mean	0.7	diff_of_gmean	9.4e-04
138	0.2	0	0	weighted_mean	0.8	diff_of_gmean	3.3e-04
138	0.2	-0.2	0.6	JKM	minMSE	—	1.4e-03
138	0.2	-0.2	0	weighted_mean	1	—	5.0e-04
138	0.15	0.5	0.6	JKM	minMSE	—	2.8e-04
138	0.15	0.5	0	weighted_mean	0.9	gmean_of_diff	8.6e-05
138	0.15	0.2	0.6	JKM	minMSE	—	3.6e-04
138	0.15	0.2	0	weighted_mean	1	—	1.3e-04
138	0.15	0	0.6	weighted_mean	0.8	diff_of_gmean	4.9e-04
138	0.15	0	0	weighted_mean	0.7	gmean_of_diff	1.8e-04
138	0.15	-0.2	0.6	JKM	minMSE	—	7.6e-04
138	0.15	-0.2	0	weighted_mean	0.8	diff_of_gmean	2.8e-04
100	0.2	0.5	0.6	JKM	minMSE	—	8.7e-04
100	0.2	0.5	0	JKM	arithmetic	—	2.3e-04
100	0.2	0.2	0.6	JKM	minMSE	—	9.7e-04
100	0.2	0.2	0	JKM	arithmetic	—	3.2e-04
100	0.2	0	0.6	weighted_mean	0.7	diff_of_gmean	1.3e-03
100	0.2	0	0	weighted_mean	0.8	diff_of_gmean	4.5e-04
100	0.2	-0.2	0.6	JKM	minMSE	—	1.9e-03
100	0.2	-0.2	0	JKM	arithmetic	—	7.1e-04
100	0.15	0.5	0.6	JKM	minMSE	—	3.9e-04
100	0.15	0.5	0	weighted_mean	0.8	diff_of_gmean	1.2e-04
100	0.15	0.2	0.6	JKM	minMSE	—	4.8e-04
100	0.15	0.2	0	JKM	minMSE	—	1.7e-04
100	0.15	0	0.6	weighted_mean	0.8	diff_of_gmean	6.8e-04
100	0.15	0	0	weighted_mean	0.7	gmean_of_diff	2.5e-04
100	0.15	-0.2	0.6	JKM	minMSE	—	1.0e-03
100	0.15	-0.2	0	weighted_mean	0.8	diff_of_gmean	3.9e-04

Table 5 Minimum MSE methods for horizon of 2.

est span	ω	γ	α	est method	est parameter	geom mean method	MSE
63	0.2	0.5	0.6	weighted_mean	0.2	gmean_of_diff	8.2e-04
63	0.2	0.5	0	weighted_mean	0.2	diff_of_gmean	3.1e-04
63	0.2	0.2	0.6	weighted_mean	0.3	diff_of_gmean	1.3e-03
63	0.2	0.2	0	weighted_mean	0.4	diff_of_gmean	4.8e-04
63	0.2	0	0.6	weighted_mean	0.3	diff_of_gmean	1.9e-03
63	0.2	0	0	weighted_mean	0.3	diff_of_gmean	7.0e-04
63	0.2	-0.2	0.6	weighted_mean	0.4	diff_of_gmean	2.9e-03
63	0.2	-0.2	0	weighted_mean	0.6	diff_of_gmean	1.1e-03
63	0.15	0.5	0.6	weighted_mean	0.2	gmean_of_diff	4.6e-04
63	0.15	0.5	0	weighted_mean	0.2	gmean_of_diff	1.8e-04
63	0.15	0.2	0.6	weighted_mean	0.3	diff_of_gmean	7.0e-04
63	0.15	0.2	0	weighted_mean	0.4	gmean_of_diff	2.8e-04
63	0.15	0	0.6	weighted_mean	0.3	diff_of_gmean	1.0e-03
63	0.15	0	0	weighted_mean	0.3	diff_of_gmean	3.9e-04
63	0.15	-0.2	0.6	weighted_mean	0.5	diff_of_gmean	1.6e-03
63	0.15	-0.2	0	weighted_mean	0.5	diff_of_gmean	6.1e-04
138	0.2	0.5	0.6	weighted_mean	0.2	gmean_of_diff	3.7e-04
138	0.2	0.5	0	weighted_mean	0.3	gmean_of_diff	1.4e-04
138	0.2	0.2	0.6	weighted_mean	0.4	gmean_of_diff	6.0e-04
138	0.2	0.2	0	weighted_mean	0.4	diff_of_gmean	2.2e-04
138	0.2	0	0.6	weighted_mean	0.3	diff_of_gmean	8.2e-04
138	0.2	0	0	weighted_mean	0.4	gmean_of_diff	3.2e-04
138	0.2	-0.2	0.6	weighted_mean	0.5	diff_of_gmean	1.3e-03
138	0.2	-0.2	0	weighted_mean	0.6	diff_of_gmean	4.9e-04
138	0.15	0.5	0.6	weighted_mean	0.2	diff_of_gmean	2.1e-04
138	0.15	0.5	0	weighted_mean	0.2	gmean_of_diff	8.0e-05
138	0.15	0.2	0.6	weighted_mean	0.4	gmean_of_diff	3.2e-04
138	0.15	0.2	0	weighted_mean	0.4	gmean_of_diff	1.2e-04
138	0.15	0	0.6	weighted_mean	0.4	gmean_of_diff	4.6e-04
138	0.15	0	0	weighted_mean	0.3	diff_of_gmean	1.8e-04
138	0.15	-0.2	0.6	weighted_mean	0.6	diff_of_gmean	7.4e-04
138	0.15	-0.2	0	weighted_mean	0.6	diff_of_gmean	2.8e-04
100	0.2	0.5	0.6	weighted_mean	0.2	gmean_of_diff	5.1e-04
100	0.2	0.5	0	weighted_mean	0.2	diff_of_gmean	1.9e-04
100	0.2	0.2	0.6	weighted_mean	0.3	diff_of_gmean	8.3e-04
100	0.2	0.2	0	weighted_mean	0.4	diff_of_gmean	3.0e-04
100	0.2	0	0.6	weighted_mean	0.3	diff_of_gmean	1.1e-03
100	0.2	0	0	weighted_mean	0.4	gmean_of_diff	4.4e-04
100	0.2	-0.2	0.6	weighted_mean	0.5	diff_of_gmean	1.8e-03
100	0.2	-0.2	0	weighted_mean	0.6	diff_of_gmean	6.9e-04
100	0.15	0.5	0.6	weighted_mean	0.2	diff_of_gmean	2.9e-04
100	0.15	0.5	0	weighted_mean	0.2	gmean_of_diff	1.1e-04
100	0.15	0.2	0.6	weighted_mean	0.4	gmean_of_diff	4.4e-04
100	0.15	0.2	0	weighted_mean	0.4	gmean_of_diff	1.7e-04
100	0.15	0	0.6	weighted_mean	0.3	diff_of_gmean	6.4e-04
100	0.15	0	0	weighted_mean	0.3	diff_of_gmean	2.4e-04
100	0.15	-0.2	0.6	weighted_mean	0.6	diff_of_gmean	1.0e-03
100	0.15	-0.2	0	weighted_mean	0.6	gmean_of_diff	3.8e-04

Table 6 Minimum MSE methods for horizon of 3.

est span	ω	γ	α	est method	est parameter	geom mean method	MSE
63	0.2	0.5	0.6	weighted_mean	0.1	gmean_of_diff	7.8e-04
63	0.2	0.5	0	weighted_mean	0.1	diff_of_gmean	3.0e-04
63	0.2	0.2	0.6	weighted_mean	0.2	diff_of_gmean	1.2e-03
63	0.2	0.2	0	weighted_mean	0.2	diff_of_gmean	4.8e-04
63	0.2	0	0.6	weighted_mean	0.2	diff_of_gmean	1.8e-03
63	0.2	0	0	weighted_mean	0.2	diff_of_gmean	7.0e-04
63	0.2	-0.2	0.6	weighted_mean	0.3	diff_of_gmean	2.8e-03
63	0.2	-0.2	0	weighted_mean	0.4	diff_of_gmean	1.1e-03
63	0.15	0.5	0.6	weighted_mean	0.1	diff_of_gmean	4.6e-04
63	0.15	0.5	0	weighted_mean	0.1	diff_of_gmean	1.7e-04
63	0.15	0.2	0.6	weighted_mean	0.2	diff_of_gmean	6.8e-04
63	0.15	0.2	0	weighted_mean	0.2	diff_of_gmean	2.8e-04
63	0.15	0	0.6	weighted_mean	0.2	diff_of_gmean	1.0e-03
63	0.15	0	0	weighted_mean	0.2	diff_of_gmean	3.9e-04
63	0.15	-0.2	0.6	weighted_mean	0.3	diff_of_gmean	1.6e-03
63	0.15	-0.2	0	weighted_mean	0.4	diff_of_gmean	6.1e-04
138	0.2	0.5	0.6	weighted_mean	0.1	diff_of_gmean	3.5e-04
138	0.2	0.5	0	weighted_mean	0.1	diff_of_gmean	1.4e-04
138	0.2	0.2	0.6	weighted_mean	0.2	diff_of_gmean	5.6e-04
138	0.2	0.2	0	weighted_mean	0.3	gmean_of_diff	2.2e-04
138	0.2	0	0.6	weighted_mean	0.2	diff_of_gmean	8.1e-04
138	0.2	0	0	weighted_mean	0.3	gmean_of_diff	3.2e-04
138	0.2	-0.2	0.6	weighted_mean	0.3	diff_of_gmean	1.3e-03
138	0.2	-0.2	0	weighted_mean	0.4	diff_of_gmean	4.8e-04
138	0.15	0.5	0.6	weighted_mean	0.1	diff_of_gmean	2.0e-04
138	0.15	0.5	0	weighted_mean	0.1	diff_of_gmean	8.0e-05
138	0.15	0.2	0.6	weighted_mean	0.2	diff_of_gmean	3.1e-04
138	0.15	0.2	0	weighted_mean	0.2	diff_of_gmean	1.2e-04
138	0.15	0	0.6	weighted_mean	0.2	diff_of_gmean	4.5e-04
138	0.15	0	0	weighted_mean	0.2	diff_of_gmean	1.8e-04
138	0.15	-0.2	0.6	weighted_mean	0.4	diff_of_gmean	7.2e-04
138	0.15	-0.2	0	weighted_mean	0.4	diff_of_gmean	2.8e-04
100	0.2	0.5	0.6	weighted_mean	0.1	gmean_of_diff	4.8e-04
100	0.2	0.5	0	weighted_mean	0.1	diff_of_gmean	1.9e-04
100	0.2	0.2	0.6	weighted_mean	0.2	diff_of_gmean	7.8e-04
100	0.2	0.2	0	weighted_mean	0.2	diff_of_gmean	3.0e-04
100	0.2	0	0.6	weighted_mean	0.2	diff_of_gmean	1.1e-03
100	0.2	0	0	weighted_mean	0.3	gmean_of_diff	4.4e-04
100	0.2	-0.2	0.6	weighted_mean	0.3	diff_of_gmean	1.7e-03
100	0.2	-0.2	0	weighted_mean	0.4	diff_of_gmean	6.8e-04
100	0.15	0.5	0.6	weighted_mean	0.1	diff_of_gmean	2.8e-04
100	0.15	0.5	0	weighted_mean	0.1	diff_of_gmean	1.1e-04
100	0.15	0.2	0.6	weighted_mean	0.2	diff_of_gmean	4.2e-04
100	0.15	0.2	0	weighted_mean	0.2	diff_of_gmean	1.7e-04
100	0.15	0	0.6	weighted_mean	0.2	diff_of_gmean	6.2e-04
100	0.15	0	0	weighted_mean	0.2	diff_of_gmean	2.4e-04
100	0.15	-0.2	0.6	weighted_mean	0.4	diff_of_gmean	9.9e-04
100	0.15	-0.2	0	weighted_mean	0.4	diff_of_gmean	3.8e-04

Table 7 Minimum MSE methods for horizon of 4.

est span	ω	γ	α	est method	est parameter	geom mean method	MSE
63	0.2	0.5	0.6	weighted_mean	0	diff_of_gmean	7.8e-04
63	0.2	0.5	0	weighted_mean	0.1	gmean_of_diff	3.0e-04
63	0.2	0.2	0.6	weighted_mean	0.1	diff_of_gmean	1.2e-03
63	0.2	0.2	0	weighted_mean	0.1	diff_of_gmean	4.7e-04
63	0.2	0	0.6	weighted_mean	0.1	diff_of_gmean	1.8e-03
63	0.2	0	0	weighted_mean	0.2	diff_of_gmean	6.9e-04
63	0.2	-0.2	0.6	weighted_mean	0.2	diff_of_gmean	2.8e-03
63	0.2	-0.2	0	weighted_mean	0.2	diff_of_gmean	1.1e-03
63	0.15	0.5	0.6	weighted_mean	0	diff_of_gmean	4.5e-04
63	0.15	0.5	0	weighted_mean	0	diff_of_gmean	1.7e-04
63	0.15	0.2	0.6	weighted_mean	0.1	diff_of_gmean	6.8e-04
63	0.15	0.2	0	weighted_mean	0.1	diff_of_gmean	2.8e-04
63	0.15	0	0.6	weighted_mean	0.1	diff_of_gmean	9.9e-04
63	0.15	0	0	weighted_mean	0.1	diff_of_gmean	3.9e-04
63	0.15	-0.2	0.6	weighted_mean	0.3	diff_of_gmean	1.6e-03
63	0.15	-0.2	0	weighted_mean	0.3	diff_of_gmean	6.0e-04
138	0.2	0.5	0.6	weighted_mean	0.1	gmean_of_diff	3.5e-04
138	0.2	0.5	0	weighted_mean	0.1	gmean_of_diff	1.4e-04
138	0.2	0.2	0.6	weighted_mean	0.2	gmean_of_diff	5.6e-04
138	0.2	0.2	0	weighted_mean	0.1	diff_of_gmean	2.2e-04
138	0.2	0	0.6	weighted_mean	0.2	diff_of_gmean	7.9e-04
138	0.2	0	0	weighted_mean	0.2	gmean_of_diff	3.1e-04
138	0.2	-0.2	0.6	weighted_mean	0.2	diff_of_gmean	1.3e-03
138	0.2	-0.2	0	weighted_mean	0.3	diff_of_gmean	4.8e-04
138	0.15	0.5	0.6	weighted_mean	0.1	gmean_of_diff	2.0e-04
138	0.15	0.5	0	weighted_mean	0	diff_of_gmean	7.9e-05
138	0.15	0.2	0.6	weighted_mean	0.2	gmean_of_diff	3.1e-04
138	0.15	0.2	0	weighted_mean	0.1	diff_of_gmean	1.2e-04
138	0.15	0	0.6	weighted_mean	0.1	diff_of_gmean	4.5e-04
138	0.15	0	0	weighted_mean	0.1	diff_of_gmean	1.8e-04
138	0.15	-0.2	0.6	weighted_mean	0.3	diff_of_gmean	7.1e-04
138	0.15	-0.2	0	weighted_mean	0.4	diff_of_gmean	2.8e-04
100	0.2	0.5	0.6	weighted_mean	0.1	gmean_of_diff	4.8e-04
100	0.2	0.5	0	weighted_mean	0.1	gmean_of_diff	1.9e-04
100	0.2	0.2	0.6	weighted_mean	0.1	diff_of_gmean	7.7e-04
100	0.2	0.2	0	weighted_mean	0.1	diff_of_gmean	3.0e-04
100	0.2	0	0.6	weighted_mean	0.2	diff_of_gmean	1.1e-03
100	0.2	0	0	weighted_mean	0.2	diff_of_gmean	4.3e-04
100	0.2	-0.2	0.6	weighted_mean	0.2	diff_of_gmean	1.7e-03
100	0.2	-0.2	0	weighted_mean	0.3	diff_of_gmean	6.8e-04
100	0.15	0.5	0.6	weighted_mean	0.1	gmean_of_diff	2.8e-04
100	0.15	0.5	0	weighted_mean	0	diff_of_gmean	1.1e-04
100	0.15	0.2	0.6	weighted_mean	0.1	diff_of_gmean	4.2e-04
100	0.15	0.2	0	weighted_mean	0.1	diff_of_gmean	1.7e-04
100	0.15	0	0.6	weighted_mean	0.1	diff_of_gmean	6.2e-04
100	0.15	0	0	weighted_mean	0.1	diff_of_gmean	2.4e-04
100	0.15	-0.2	0.6	weighted_mean	0.3	diff_of_gmean	9.8e-04
100	0.15	-0.2	0	weighted_mean	0.3	diff_of_gmean	3.8e-04

Table 8 Minimum MSE methods for horizon of 5.

est span	ω	γ	α	est method	est parameter	geom mean method	MSE
63	0.2	0.5	0.6	weighted_mean	0	diff_of_gmean	7.6e-04
63	0.2	0.5	0	weighted_mean	0.1	gmean_of_diff	3.0e-04
63	0.2	0.2	0.6	weighted_mean	0.1	diff_of_gmean	1.2e-03
63	0.2	0.2	0	weighted_mean	0.1	diff_of_gmean	4.7e-04
63	0.2	0	0.6	weighted_mean	0.1	diff_of_gmean	1.8e-03
63	0.2	0	0	weighted_mean	0.1	diff_of_gmean	6.9e-04
63	0.2	-0.2	0.6	weighted_mean	0.1	diff_of_gmean	2.7e-03
63	0.2	-0.2	0	weighted_mean	0.2	diff_of_gmean	1.1e-03
63	0.15	0.5	0.6	weighted_mean	0	diff_of_gmean	4.5e-04
63	0.15	0.5	0	weighted_mean	0	diff_of_gmean	1.7e-04
63	0.15	0.2	0.6	weighted_mean	0.1	diff_of_gmean	6.7e-04
63	0.15	0.2	0	weighted_mean	0.1	gmean_of_diff	2.8e-04
63	0.15	0	0.6	weighted_mean	0	diff_of_gmean	9.8e-04
63	0.15	0	0	weighted_mean	0.1	diff_of_gmean	3.9e-04
63	0.15	-0.2	0.6	weighted_mean	0.2	diff_of_gmean	1.5e-03
63	0.15	-0.2	0	weighted_mean	0.3	diff_of_gmean	6.0e-04
138	0.2	0.5	0.6	weighted_mean	0	diff_of_gmean	3.4e-04
138	0.2	0.5	0	weighted_mean	0.1	gmean_of_diff	1.4e-04
138	0.2	0.2	0.6	weighted_mean	0.1	diff_of_gmean	5.4e-04
138	0.2	0.2	0	weighted_mean	0.1	diff_of_gmean	2.2e-04
138	0.2	0	0.6	weighted_mean	0.1	diff_of_gmean	7.8e-04
138	0.2	0	0	weighted_mean	0.1	diff_of_gmean	3.1e-04
138	0.2	-0.2	0.6	weighted_mean	0.2	diff_of_gmean	1.2e-03
138	0.2	-0.2	0	weighted_mean	0.2	diff_of_gmean	4.8e-04
138	0.15	0.5	0.6	weighted_mean	0.1	gmean_of_diff	2.0e-04
138	0.15	0.5	0	weighted_mean	0	diff_of_gmean	7.9e-05
138	0.15	0.2	0.6	weighted_mean	0.1	diff_of_gmean	3.1e-04
138	0.15	0.2	0	weighted_mean	0.1	diff_of_gmean	1.2e-04
138	0.15	0	0.6	weighted_mean	0.1	gmean_of_diff	4.4e-04
138	0.15	0	0	weighted_mean	0.1	diff_of_gmean	1.8e-04
138	0.15	-0.2	0.6	weighted_mean	0.2	diff_of_gmean	7.0e-04
138	0.15	-0.2	0	weighted_mean	0.3	diff_of_gmean	2.8e-04
100	0.2	0.5	0.6	weighted_mean	0	diff_of_gmean	4.7e-04
100	0.2	0.5	0	weighted_mean	0.1	gmean_of_diff	1.9e-04
100	0.2	0.2	0.6	weighted_mean	0.1	diff_of_gmean	7.6e-04
100	0.2	0.2	0	weighted_mean	0.1	diff_of_gmean	3.0e-04
100	0.2	0	0.6	weighted_mean	0.1	diff_of_gmean	1.1e-03
100	0.2	0	0	weighted_mean	0.1	diff_of_gmean	4.3e-04
100	0.2	-0.2	0.6	weighted_mean	0.2	diff_of_gmean	1.7e-03
100	0.2	-0.2	0	weighted_mean	0.2	diff_of_gmean	6.7e-04
100	0.15	0.5	0.6	weighted_mean	0	diff_of_gmean	2.8e-04
100	0.15	0.5	0	weighted_mean	0	diff_of_gmean	1.1e-04
100	0.15	0.2	0.6	weighted_mean	0.1	diff_of_gmean	4.2e-04
100	0.15	0.2	0	weighted_mean	0.1	gmean_of_diff	1.7e-04
100	0.15	0	0.6	weighted_mean	0	diff_of_gmean	6.1e-04
100	0.15	0	0	weighted_mean	0.1	diff_of_gmean	2.4e-04
100	0.15	-0.2	0.6	weighted_mean	0.2	diff_of_gmean	9.7e-04
100	0.15	-0.2	0	weighted_mean	0.3	diff_of_gmean	3.8e-04

Table 9 Minimum MSE methods for horizon of 10.

est span	ω	γ	α	est method	est parameter	geom mean method	MSE
63	0.2	0.5	0.6	weighted_mean	0	diff_of_gmean	7.6e-04
63	0.2	0.5	0	weighted_mean	0	gmean_of_diff	3.0e-04
63	0.2	0.2	0.6	weighted_mean	0	diff_of_gmean	1.2e-03
63	0.2	0.2	0	weighted_mean	0	diff_of_gmean	4.7e-04
63	0.2	0	0.6	weighted_mean	0	diff_of_gmean	1.7e-03
63	0.2	0	0	weighted_mean	0	diff_of_gmean	6.9e-04
63	0.2	-0.2	0.6	weighted_mean	0	diff_of_gmean	2.7e-03
63	0.2	-0.2	0	weighted_mean	0.1	diff_of_gmean	1.1e-03
63	0.15	0.5	0.6	weighted_mean	0	diff_of_gmean	4.5e-04
63	0.15	0.5	0	weighted_mean	0	gmean_of_diff	1.7e-04
63	0.15	0.2	0.6	weighted_mean	0	diff_of_gmean	6.7e-04
63	0.15	0.2	0	weighted_mean	0	diff_of_gmean	2.7e-04
63	0.15	0	0.6	weighted_mean	0	diff_of_gmean	9.8e-04
63	0.15	0	0	weighted_mean	0	diff_of_gmean	3.9e-04
63	0.15	-0.2	0.6	weighted_mean	0	diff_of_gmean	1.5e-03
63	0.15	-0.2	0	weighted_mean	0.1	diff_of_gmean	6.0e-04
138	0.2	0.5	0.6	weighted_mean	0	diff_of_gmean	3.4e-04
138	0.2	0.5	0	weighted_mean	0	gmean_of_diff	1.4e-04
138	0.2	0.2	0.6	weighted_mean	0	diff_of_gmean	5.4e-04
138	0.2	0.2	0	weighted_mean	0	diff_of_gmean	2.1e-04
138	0.2	0	0.6	weighted_mean	0	diff_of_gmean	7.6e-04
138	0.2	0	0	weighted_mean	0	diff_of_gmean	3.1e-04
138	0.2	-0.2	0.6	weighted_mean	0.1	diff_of_gmean	1.2e-03
138	0.2	-0.2	0	weighted_mean	0.1	diff_of_gmean	4.8e-04
138	0.15	0.5	0.6	weighted_mean	0	diff_of_gmean	2.0e-04
138	0.15	0.5	0	weighted_mean	0	gmean_of_diff	7.9e-05
138	0.15	0.2	0.6	weighted_mean	0	diff_of_gmean	3.1e-04
138	0.15	0.2	0	weighted_mean	0	diff_of_gmean	1.2e-04
138	0.15	0	0.6	weighted_mean	0	diff_of_gmean	4.4e-04
138	0.15	0	0	weighted_mean	0	diff_of_gmean	1.8e-04
138	0.15	-0.2	0.6	weighted_mean	0.1	diff_of_gmean	6.9e-04
138	0.15	-0.2	0	weighted_mean	0.2	diff_of_gmean	2.8e-04
100	0.2	0.5	0.6	weighted_mean	0	diff_of_gmean	4.7e-04
100	0.2	0.5	0	weighted_mean	0	gmean_of_diff	1.9e-04
100	0.2	0.2	0.6	weighted_mean	0	diff_of_gmean	7.4e-04
100	0.2	0.2	0	weighted_mean	0	diff_of_gmean	3.0e-04
100	0.2	0	0.6	weighted_mean	0	diff_of_gmean	1.0e-03
100	0.2	0	0	weighted_mean	0	diff_of_gmean	4.3e-04
100	0.2	-0.2	0.6	weighted_mean	0	diff_of_gmean	1.7e-03
100	0.2	-0.2	0	weighted_mean	0.1	diff_of_gmean	6.7e-04
100	0.15	0.5	0.6	weighted_mean	0	diff_of_gmean	2.8e-04
100	0.15	0.5	0	weighted_mean	0	gmean_of_diff	1.1e-04
100	0.15	0.2	0.6	weighted_mean	0	diff_of_gmean	4.1e-04
100	0.15	0.2	0	weighted_mean	0	diff_of_gmean	1.7e-04
100	0.15	0	0.6	weighted_mean	0	diff_of_gmean	6.1e-04
100	0.15	0	0	weighted_mean	0	diff_of_gmean	2.4e-04
100	0.15	-0.2	0.6	weighted_mean	0.1	diff_of_gmean	9.5e-04
100	0.15	-0.2	0	weighted_mean	0.1	diff_of_gmean	3.8e-04

5.2 Best methods for MSE with respect to cumulative estimate over horizon

Table 10 Minimum MSE methods for horizon of 1.

est span	ω	γ	α	est method	est parameter	geom mean method	MSE
63	0.2	0.5	0.6	JKM	minMSE	——	1.4e-03
63	0.2	0.5	0	JKM	unbiased	——	3.7e-04
63	0.2	0.2	0.6	JKM	minMSE	——	1.5e-03
63	0.2	0.2	0	JKM	arithmetic	——	5.1e-04
63	0.2	0	0.6	weighted_mean	0.6	diff_of_gmean	2.1e-03
63	0.2	0	0	weighted_mean	0.8	diff_of_gmean	7.2e-04
63	0.2	-0.2	0.6	JKM	minMSE	——	3.1e-03
63	0.2	-0.2	0	JKM	minMSE	——	1.1e-03
63	0.15	0.5	0.6	JKM	minMSE	——	6.2e-04
63	0.15	0.5	0	weighted_mean	0.8	diff_of_gmean	1.9e-04
63	0.15	0.2	0.6	JKM	minMSE	——	7.8e-04
63	0.15	0.2	0	JKM	minMSE	——	2.9e-04
63	0.15	0	0.6	weighted_mean	0.8	diff_of_gmean	1.1e-03
63	0.15	0	0	weighted_mean	0.7	diff_of_gmean	3.9e-04
63	0.15	-0.2	0.6	JKM	minMSE	——	1.7e-03
63	0.15	-0.2	0	weighted_mean	0.8	diff_of_gmean	6.1e-04
138	0.2	0.5	0.6	JKM	minMSE	——	6.4e-04
138	0.2	0.5	0	JKM	arithmetic	——	1.7e-04
138	0.2	0.2	0.6	JKM	minMSE	——	6.9e-04
138	0.2	0.2	0	JKM	arithmetic	——	2.3e-04
138	0.2	0	0.6	weighted_mean	0.7	diff_of_gmean	9.4e-04
138	0.2	0	0	weighted_mean	0.8	diff_of_gmean	3.3e-04
138	0.2	-0.2	0.6	JKM	minMSE	——	1.4e-03
138	0.2	-0.2	0	weighted_mean	1	——	5.0e-04
138	0.15	0.5	0.6	JKM	minMSE	——	2.8e-04
138	0.15	0.5	0	weighted_mean	0.9	gmean_of_diff	8.6e-05
138	0.15	0.2	0.6	JKM	minMSE	——	3.6e-04
138	0.15	0.2	0	weighted_mean	1	——	1.3e-04
138	0.15	0	0.6	weighted_mean	0.8	diff_of_gmean	4.9e-04
138	0.15	0	0	weighted_mean	0.7	gmean_of_diff	1.8e-04
138	0.15	-0.2	0.6	JKM	minMSE	——	7.6e-04
138	0.15	-0.2	0	weighted_mean	0.8	diff_of_gmean	2.8e-04
100	0.2	0.5	0.6	JKM	minMSE	——	8.7e-04
100	0.2	0.5	0	JKM	arithmetic	——	2.3e-04
100	0.2	0.2	0.6	JKM	minMSE	——	9.7e-04
100	0.2	0.2	0	JKM	arithmetic	——	3.2e-04
100	0.2	0	0.6	weighted_mean	0.7	diff_of_gmean	1.3e-03
100	0.2	0	0	weighted_mean	0.8	diff_of_gmean	4.5e-04
100	0.2	-0.2	0.6	JKM	minMSE	——	1.9e-03
100	0.2	-0.2	0	JKM	arithmetic	——	7.1e-04
100	0.15	0.5	0.6	JKM	minMSE	——	3.9e-04
100	0.15	0.5	0	weighted_mean	0.8	diff_of_gmean	1.2e-04
100	0.15	0.2	0.6	JKM	minMSE	——	4.8e-04
100	0.15	0.2	0	JKM	minMSE	——	1.7e-04
100	0.15	0	0.6	weighted_mean	0.8	diff_of_gmean	6.8e-04
100	0.15	0	0	weighted_mean	0.7	gmean_of_diff	2.5e-04
100	0.15	-0.2	0.6	JKM	minMSE	——	1.0e-03
100	0.15	-0.2	0	weighted_mean	0.8	diff_of_gmean	3.9e-04

Table 11 Minimum MSE methods for horizon of 2.

est span	ω	γ	α	est method	est parameter	geom mean method	MSE
63	0.2	0.5	0.6	weighted_mean	0.1	diff_of_gmean	3.5e-03
63	0.2	0.5	0	weighted_mean	0.2	diff_of_gmean	1.3e-03
63	0.2	0.2	0.6	weighted_mean	0.3	diff_of_gmean	5.9e-03
63	0.2	0.2	0	weighted_mean	0.4	diff_of_gmean	2.1e-03
63	0.2	0	0.6	weighted_mean	0.2	diff_of_gmean	8.1e-03
63	0.2	0	0	weighted_mean	0.3	diff_of_gmean	3.0e-03
63	0.2	-0.2	0.6	weighted_mean	0.4	diff_of_gmean	1.3e-02
63	0.2	-0.2	0	weighted_mean	0.5	diff_of_gmean	4.8e-03
63	0.15	0.5	0.6	weighted_mean	0.2	gmean_of_diff	2.0e-03
63	0.15	0.5	0	weighted_mean	0.1	diff_of_gmean	7.6e-04
63	0.15	0.2	0.6	weighted_mean	0.3	diff_of_gmean	3.1e-03
63	0.15	0.2	0	weighted_mean	0.3	diff_of_gmean	1.2e-03
63	0.15	0	0.6	weighted_mean	0.3	diff_of_gmean	4.5e-03
63	0.15	0	0	weighted_mean	0.2	diff_of_gmean	1.7e-03
63	0.15	-0.2	0.6	weighted_mean	0.4	diff_of_gmean	7.1e-03
63	0.15	-0.2	0	weighted_mean	0.5	diff_of_gmean	2.7e-03
138	0.2	0.5	0.6	weighted_mean	0.2	gmean_of_diff	1.6e-03
138	0.2	0.5	0	weighted_mean	0.2	diff_of_gmean	6.0e-04
138	0.2	0.2	0.6	weighted_mean	0.3	diff_of_gmean	2.6e-03
138	0.2	0.2	0	weighted_mean	0.4	diff_of_gmean	9.5e-04
138	0.2	0	0.6	weighted_mean	0.3	diff_of_gmean	3.6e-03
138	0.2	0	0	weighted_mean	0.3	diff_of_gmean	1.4e-03
138	0.2	-0.2	0.6	weighted_mean	0.5	diff_of_gmean	6.0e-03
138	0.2	-0.2	0	weighted_mean	0.6	diff_of_gmean	2.1e-03
138	0.15	0.5	0.6	weighted_mean	0.2	diff_of_gmean	9.0e-04
138	0.15	0.5	0	weighted_mean	0.2	gmean_of_diff	3.5e-04
138	0.15	0.2	0.6	weighted_mean	0.4	gmean_of_diff	1.4e-03
138	0.15	0.2	0	weighted_mean	0.4	gmean_of_diff	5.4e-04
138	0.15	0	0.6	weighted_mean	0.3	diff_of_gmean	2.0e-03
138	0.15	0	0	weighted_mean	0.3	diff_of_gmean	7.7e-04
138	0.15	-0.2	0.6	weighted_mean	0.6	diff_of_gmean	3.3e-03
138	0.15	-0.2	0	weighted_mean	0.6	gmean_of_diff	1.2e-03
100	0.2	0.5	0.6	weighted_mean	0.2	gmean_of_diff	2.2e-03
100	0.2	0.5	0	weighted_mean	0.2	diff_of_gmean	8.3e-04
100	0.2	0.2	0.6	weighted_mean	0.3	diff_of_gmean	3.7e-03
100	0.2	0.2	0	weighted_mean	0.4	diff_of_gmean	1.3e-03
100	0.2	0	0.6	weighted_mean	0.3	diff_of_gmean	5.0e-03
100	0.2	0	0	weighted_mean	0.3	diff_of_gmean	1.9e-03
100	0.2	-0.2	0.6	weighted_mean	0.4	diff_of_gmean	8.1e-03
100	0.2	-0.2	0	weighted_mean	0.6	gmean_of_diff	3.0e-03
100	0.15	0.5	0.6	weighted_mean	0.2	gmean_of_diff	1.3e-03
100	0.15	0.5	0	weighted_mean	0.2	gmean_of_diff	4.8e-04
100	0.15	0.2	0.6	weighted_mean	0.3	diff_of_gmean	1.9e-03
100	0.15	0.2	0	weighted_mean	0.4	gmean_of_diff	7.3e-04
100	0.15	0	0.6	weighted_mean	0.3	diff_of_gmean	2.8e-03
100	0.15	0	0	weighted_mean	0.3	diff_of_gmean	1.1e-03
100	0.15	-0.2	0.6	weighted_mean	0.5	diff_of_gmean	4.5e-03
100	0.15	-0.2	0	weighted_mean	0.5	diff_of_gmean	1.7e-03

Table 12 Minimum MSE methods for horizon of 3.

est span	ω	γ	α	est method	est parameter	geom mean method	MSE
63	0.2	0.5	0.6	weighted_mean	0	diff_of_gmean	8.1e-03
63	0.2	0.5	0	weighted_mean	0.1	gmean_of_diff	3.1e-03
63	0.2	0.2	0.6	weighted_mean	0.1	diff_of_gmean	1.3e-02
63	0.2	0.2	0	weighted_mean	0.2	gmean_of_diff	4.9e-03
63	0.2	0	0.6	weighted_mean	0.1	diff_of_gmean	1.9e-02
63	0.2	0	0	weighted_mean	0.1	diff_of_gmean	7.2e-03
63	0.2	-0.2	0.6	weighted_mean	0.1	diff_of_gmean	3.1e-02
63	0.2	-0.2	0	weighted_mean	0.2	diff_of_gmean	1.1e-02
63	0.15	0.5	0.6	weighted_mean	0.1	diff_of_gmean	4.9e-03
63	0.15	0.5	0	weighted_mean	0.1	gmean_of_diff	1.8e-03
63	0.15	0.2	0.6	weighted_mean	0.1	diff_of_gmean	7.3e-03
63	0.15	0.2	0	weighted_mean	0.1	diff_of_gmean	2.9e-03
63	0.15	0	0.6	weighted_mean	0.1	diff_of_gmean	1.1e-02
63	0.15	0	0	weighted_mean	0.1	diff_of_gmean	4.1e-03
63	0.15	-0.2	0.6	weighted_mean	0.2	diff_of_gmean	1.7e-02
63	0.15	-0.2	0	weighted_mean	0.2	diff_of_gmean	6.4e-03
138	0.2	0.5	0.6	weighted_mean	0.1	gmean_of_diff	3.7e-03
138	0.2	0.5	0	weighted_mean	0.1	diff_of_gmean	1.4e-03
138	0.2	0.2	0.6	weighted_mean	0.2	diff_of_gmean	6.0e-03
138	0.2	0.2	0	weighted_mean	0.2	diff_of_gmean	2.2e-03
138	0.2	0	0.6	weighted_mean	0.2	diff_of_gmean	8.6e-03
138	0.2	0	0	weighted_mean	0.2	diff_of_gmean	3.3e-03
138	0.2	-0.2	0.6	weighted_mean	0.3	diff_of_gmean	1.4e-02
138	0.2	-0.2	0	weighted_mean	0.3	diff_of_gmean	5.0e-03
138	0.15	0.5	0.6	weighted_mean	0.1	diff_of_gmean	2.2e-03
138	0.15	0.5	0	weighted_mean	0.1	diff_of_gmean	8.4e-04
138	0.15	0.2	0.6	weighted_mean	0.2	diff_of_gmean	3.4e-03
138	0.15	0.2	0	weighted_mean	0.2	diff_of_gmean	1.3e-03
138	0.15	0	0.6	weighted_mean	0.2	diff_of_gmean	4.9e-03
138	0.15	0	0	weighted_mean	0.2	gmean_of_diff	1.9e-03
138	0.15	-0.2	0.6	weighted_mean	0.4	diff_of_gmean	8.0e-03
138	0.15	-0.2	0	weighted_mean	0.4	gmean_of_diff	3.0e-03
100	0.2	0.5	0.6	weighted_mean	0.1	gmean_of_diff	5.0e-03
100	0.2	0.5	0	weighted_mean	0.1	gmean_of_diff	1.9e-03
100	0.2	0.2	0.6	weighted_mean	0.1	diff_of_gmean	8.4e-03
100	0.2	0.2	0	weighted_mean	0.2	diff_of_gmean	3.1e-03
100	0.2	0	0.6	weighted_mean	0.2	diff_of_gmean	1.2e-02
100	0.2	0	0	weighted_mean	0.2	diff_of_gmean	4.5e-03
100	0.2	-0.2	0.6	weighted_mean	0.2	diff_of_gmean	1.9e-02
100	0.2	-0.2	0	weighted_mean	0.3	diff_of_gmean	7.1e-03
100	0.15	0.5	0.6	weighted_mean	0.1	diff_of_gmean	3.0e-03
100	0.15	0.5	0	weighted_mean	0.1	gmean_of_diff	1.2e-03
100	0.15	0.2	0.6	weighted_mean	0.2	diff_of_gmean	4.6e-03
100	0.15	0.2	0	weighted_mean	0.2	gmean_of_diff	1.8e-03
100	0.15	0	0.6	weighted_mean	0.1	diff_of_gmean	6.7e-03
100	0.15	0	0	weighted_mean	0.1	diff_of_gmean	2.6e-03
100	0.15	-0.2	0.6	weighted_mean	0.3	diff_of_gmean	1.1e-02
100	0.15	-0.2	0	weighted_mean	0.3	diff_of_gmean	4.1e-03

Table 13 Minimum MSE methods for horizon of 4.

est span	ω	γ	α	est method	est parameter	geom mean method	MSE
63	0.2	0.5	0.6	weighted_mean	0	diff_of_gmean	1.5e-02
63	0.2	0.5	0	weighted_mean	0	diff_of_gmean	5.9e-03
63	0.2	0.2	0.6	weighted_mean	0	diff_of_gmean	2.4e-02
63	0.2	0.2	0	weighted_mean	0	diff_of_gmean	9.2e-03
63	0.2	0	0.6	weighted_mean	0	diff_of_gmean	3.6e-02
63	0.2	0	0	weighted_mean	0	diff_of_gmean	1.3e-02
63	0.2	-0.2	0.6	weighted_mean	0	diff_of_gmean	5.7e-02
63	0.2	-0.2	0	weighted_mean	0	diff_of_gmean	2.1e-02
63	0.15	0.5	0.6	weighted_mean	0	diff_of_gmean	9.2e-03
63	0.15	0.5	0	weighted_mean	0	gmean_of_diff	3.5e-03
63	0.15	0.2	0.6	weighted_mean	0	diff_of_gmean	1.4e-02
63	0.15	0.2	0	weighted_mean	0	diff_of_gmean	5.6e-03
63	0.15	0	0.6	weighted_mean	0	gmean_of_diff	2.0e-02
63	0.15	0	0	weighted_mean	0	diff_of_gmean	7.8e-03
63	0.15	-0.2	0.6	weighted_mean	0	diff_of_gmean	3.3e-02
63	0.15	-0.2	0	weighted_mean	0.1	diff_of_gmean	1.2e-02
138	0.2	0.5	0.6	weighted_mean	0	diff_of_gmean	6.9e-03
138	0.2	0.5	0	weighted_mean	0.1	gmean_of_diff	2.7e-03
138	0.2	0.2	0.6	weighted_mean	0.1	diff_of_gmean	1.1e-02
138	0.2	0.2	0	weighted_mean	0.1	diff_of_gmean	4.2e-03
138	0.2	0	0.6	weighted_mean	0.1	diff_of_gmean	1.6e-02
138	0.2	0	0	weighted_mean	0.1	diff_of_gmean	6.1e-03
138	0.2	-0.2	0.6	weighted_mean	0.2	diff_of_gmean	2.6e-02
138	0.2	-0.2	0	weighted_mean	0.2	diff_of_gmean	9.5e-03
138	0.15	0.5	0.6	weighted_mean	0	diff_of_gmean	4.1e-03
138	0.15	0.5	0	weighted_mean	0	diff_of_gmean	1.6e-03
138	0.15	0.2	0.6	weighted_mean	0.1	diff_of_gmean	6.4e-03
138	0.15	0.2	0	weighted_mean	0.1	gmean_of_diff	2.5e-03
138	0.15	0	0.6	weighted_mean	0	diff_of_gmean	9.2e-03
138	0.15	0	0	weighted_mean	0.1	gmean_of_diff	3.6e-03
138	0.15	-0.2	0.6	weighted_mean	0.2	diff_of_gmean	1.5e-02
138	0.15	-0.2	0	weighted_mean	0.3	diff_of_gmean	5.7e-03
100	0.2	0.5	0.6	weighted_mean	0	diff_of_gmean	9.3e-03
100	0.2	0.5	0	weighted_mean	0	diff_of_gmean	3.7e-03
100	0.2	0.2	0.6	weighted_mean	0.1	diff_of_gmean	1.5e-02
100	0.2	0.2	0	weighted_mean	0.1	gmean_of_diff	5.8e-03
100	0.2	0	0.6	weighted_mean	0.1	diff_of_gmean	2.2e-02
100	0.2	0	0	weighted_mean	0.1	diff_of_gmean	8.4e-03
100	0.2	-0.2	0.6	weighted_mean	0.1	diff_of_gmean	3.6e-02
100	0.2	-0.2	0	weighted_mean	0.1	diff_of_gmean	1.3e-02
100	0.15	0.5	0.6	weighted_mean	0	diff_of_gmean	5.7e-03
100	0.15	0.5	0	weighted_mean	0	diff_of_gmean	2.2e-03
100	0.15	0.2	0.6	weighted_mean	0.1	diff_of_gmean	8.7e-03
100	0.15	0.2	0	weighted_mean	0	diff_of_gmean	3.4e-03
100	0.15	0	0.6	weighted_mean	0	diff_of_gmean	1.3e-02
100	0.15	0	0	weighted_mean	0.1	gmean_of_diff	4.9e-03
100	0.15	-0.2	0.6	weighted_mean	0.2	diff_of_gmean	2.1e-02
100	0.15	-0.2	0	weighted_mean	0.2	diff_of_gmean	7.8e-03

Table 14 Minimum MSE methods for horizon of 5.

est span	ω	γ	α	est method	est parameter	geom mean method	MSE
63	0.2	0.5	0.6	weighted_mean	0	gmean_of_diff	2.5e-02
63	0.2	0.5	0	weighted_mean	0	diff_of_gmean	9.7e-03
63	0.2	0.2	0.6	weighted_mean	0	diff_of_gmean	4.0e-02
63	0.2	0.2	0	weighted_mean	0	diff_of_gmean	1.5e-02
63	0.2	0	0.6	weighted_mean	0	gmean_of_diff	5.8e-02
63	0.2	0	0	weighted_mean	0	gmean_of_diff	2.2e-02
63	0.2	-0.2	0.6	weighted_mean	0	gmean_of_diff	9.4e-02
63	0.2	-0.2	0	weighted_mean	0	gmean_of_diff	3.5e-02
63	0.15	0.5	0.6	weighted_mean	0	gmean_of_diff	1.6e-02
63	0.15	0.5	0	weighted_mean	0	gmean_of_diff	5.9e-03
63	0.15	0.2	0.6	weighted_mean	0	gmean_of_diff	2.3e-02
63	0.15	0.2	0	weighted_mean	0	gmean_of_diff	9.4e-03
63	0.15	0	0.6	weighted_mean	0	gmean_of_diff	3.4e-02
63	0.15	0	0	weighted_mean	0	gmean_of_diff	1.3e-02
63	0.15	-0.2	0.6	weighted_mean	0	gmean_of_diff	5.5e-02
63	0.15	-0.2	0	weighted_mean	0	diff_of_gmean	2.1e-02
138	0.2	0.5	0.6	weighted_mean	0	diff_of_gmean	1.1e-02
138	0.2	0.5	0	weighted_mean	0	diff_of_gmean	4.4e-03
138	0.2	0.2	0.6	weighted_mean	0.1	gmean_of_diff	1.8e-02
138	0.2	0.2	0	weighted_mean	0.1	diff_of_gmean	7.0e-03
138	0.2	0	0.6	weighted_mean	0	diff_of_gmean	2.6e-02
138	0.2	0	0	weighted_mean	0	diff_of_gmean	1.0e-02
138	0.2	-0.2	0.6	weighted_mean	0	diff_of_gmean	4.3e-02
138	0.2	-0.2	0	weighted_mean	0.1	diff_of_gmean	1.6e-02
138	0.15	0.5	0.6	weighted_mean	0	diff_of_gmean	6.9e-03
138	0.15	0.5	0	weighted_mean	0	diff_of_gmean	2.7e-03
138	0.15	0.2	0.6	weighted_mean	0	diff_of_gmean	1.1e-02
138	0.15	0.2	0	weighted_mean	0	diff_of_gmean	4.2e-03
138	0.15	0	0.6	weighted_mean	0	diff_of_gmean	1.5e-02
138	0.15	0	0	weighted_mean	0	diff_of_gmean	6.0e-03
138	0.15	-0.2	0.6	weighted_mean	0.1	diff_of_gmean	2.5e-02
138	0.15	-0.2	0	weighted_mean	0.2	diff_of_gmean	9.7e-03
100	0.2	0.5	0.6	weighted_mean	0	diff_of_gmean	1.5e-02
100	0.2	0.5	0	weighted_mean	0	diff_of_gmean	6.1e-03
100	0.2	0.2	0.6	weighted_mean	0	diff_of_gmean	2.5e-02
100	0.2	0.2	0	weighted_mean	0.1	gmean_of_diff	9.7e-03
100	0.2	0	0.6	weighted_mean	0	diff_of_gmean	3.5e-02
100	0.2	0	0	weighted_mean	0	diff_of_gmean	1.4e-02
100	0.2	-0.2	0.6	weighted_mean	0	diff_of_gmean	5.8e-02
100	0.2	-0.2	0	weighted_mean	0	diff_of_gmean	2.2e-02
100	0.15	0.5	0.6	weighted_mean	0	diff_of_gmean	9.6e-03
100	0.15	0.5	0	weighted_mean	0	gmean_of_diff	3.7e-03
100	0.15	0.2	0.6	weighted_mean	0	diff_of_gmean	1.4e-02
100	0.15	0.2	0	weighted_mean	0	diff_of_gmean	5.7e-03
100	0.15	0	0.6	weighted_mean	0	gmean_of_diff	2.1e-02
100	0.15	0	0	weighted_mean	0	diff_of_gmean	8.3e-03
100	0.15	-0.2	0.6	weighted_mean	0	diff_of_gmean	3.5e-02
100	0.15	-0.2	0	weighted_mean	0.1	diff_of_gmean	1.3e-02

Table 15 Minimum MSE methods for horizon of 10.

est span	ω	γ	α	est method	est parameter	geom mean method	MSE
63	0.2	0.5	0.6	weighted_mean	0	gmean_of_diff	1.4e-01
63	0.2	0.5	0	weighted_mean	0	gmean_of_diff	5.3e-02
63	0.2	0.2	0.6	weighted_mean	0	gmean_of_diff	2.3e-01
63	0.2	0.2	0	weighted_mean	0	gmean_of_diff	8.4e-02
63	0.2	0	0.6	weighted_mean	0	gmean_of_diff	3.8e-01
63	0.2	0	0	weighted_mean	0	gmean_of_diff	1.3e-01
63	0.2	-0.2	0.6	weighted_mean	0	gmean_of_diff	6.7e-01
63	0.2	-0.2	0	weighted_mean	0	gmean_of_diff	2.1e-01
63	0.15	0.5	0.6	weighted_mean	0	gmean_of_diff	9.5e-02
63	0.15	0.5	0	weighted_mean	0	gmean_of_diff	3.5e-02
63	0.15	0.2	0.6	weighted_mean	0	gmean_of_diff	1.4e-01
63	0.15	0.2	0	weighted_mean	0	gmean_of_diff	5.7e-02
63	0.15	0	0.6	weighted_mean	0	gmean_of_diff	2.2e-01
63	0.15	0	0	weighted_mean	0	gmean_of_diff	7.9e-02
63	0.15	-0.2	0.6	weighted_mean	0	gmean_of_diff	3.7e-01
63	0.15	-0.2	0	weighted_mean	0	gmean_of_diff	1.3e-01
138	0.2	0.5	0.6	weighted_mean	0	gmean_of_diff	6.0e-02
138	0.2	0.5	0	weighted_mean	0	gmean_of_diff	2.3e-02
138	0.2	0.2	0.6	weighted_mean	0	gmean_of_diff	9.6e-02
138	0.2	0.2	0	weighted_mean	0	gmean_of_diff	3.7e-02
138	0.2	0	0.6	weighted_mean	0	gmean_of_diff	1.4e-01
138	0.2	0	0	weighted_mean	0	gmean_of_diff	5.5e-02
138	0.2	-0.2	0.6	weighted_mean	0	gmean_of_diff	2.4e-01
138	0.2	-0.2	0	weighted_mean	0	gmean_of_diff	8.5e-02
138	0.15	0.5	0.6	weighted_mean	0	gmean_of_diff	4.0e-02
138	0.15	0.5	0	weighted_mean	0	gmean_of_diff	1.6e-02
138	0.15	0.2	0.6	weighted_mean	0	gmean_of_diff	6.2e-02
138	0.15	0.2	0	weighted_mean	0	gmean_of_diff	2.4e-02
138	0.15	0	0.6	weighted_mean	0	gmean_of_diff	9.3e-02
138	0.15	0	0	weighted_mean	0	gmean_of_diff	3.5e-02
138	0.15	-0.2	0.6	weighted_mean	0	gmean_of_diff	1.5e-01
138	0.15	-0.2	0	weighted_mean	0	gmean_of_diff	5.6e-02
100	0.2	0.5	0.6	weighted_mean	0	gmean_of_diff	8.3e-02
100	0.2	0.5	0	weighted_mean	0	gmean_of_diff	3.2e-02
100	0.2	0.2	0.6	weighted_mean	0	gmean_of_diff	1.4e-01
100	0.2	0.2	0	weighted_mean	0	gmean_of_diff	5.2e-02
100	0.2	0	0.6	weighted_mean	0	gmean_of_diff	2.1e-01
100	0.2	0	0	weighted_mean	0	gmean_of_diff	7.6e-02
100	0.2	-0.2	0.6	weighted_mean	0	gmean_of_diff	3.5e-01
100	0.2	-0.2	0	weighted_mean	0	gmean_of_diff	1.2e-01
100	0.15	0.5	0.6	weighted_mean	0	gmean_of_diff	5.6e-02
100	0.15	0.5	0	weighted_mean	0	gmean_of_diff	2.2e-02
100	0.15	0.2	0.6	weighted_mean	0	gmean_of_diff	8.6e-02
100	0.15	0.2	0	weighted_mean	0	gmean_of_diff	3.4e-02
100	0.15	0	0.6	weighted_mean	0	gmean_of_diff	1.3e-01
100	0.15	0	0	weighted_mean	0	gmean_of_diff	4.9e-02
100	0.15	-0.2	0.6	weighted_mean	0	gmean_of_diff	2.1e-01
100	0.15	-0.2	0	weighted_mean	0	gmean_of_diff	7.8e-02